

**Conformal Anomaly via AdS/CFT Duality \***SACHIKO OGUSHI<sup>†</sup>*Department of Physics, Ochanomizu University  
Otsuka, Bunkyo-ku Tokyo 112-0012, JAPAN***ABSTRACT**

AdS/CFT duality is a conjectured dual correspondence between the large  $N$  limit of Conformal Field Theory (CFT) in  $d$ -dimensions and the supergravity (SUGRA) in  $d+1$ -dimensional Anti de Sitter (AdS) space. By using this conjecture, we can study various properties of large  $N$  CFT by simple calculations in SUGRA. Recently much attention has been paid to the Renormalization Group (RG) flow viewed from the SUGRA side. Such RG flow in CFT is known to be characterized by the  $c$ -function which connects CFTs with different central charges. Therefore, we are interested in deriving this  $c$ -function from SUGRA with the help of AdS/CFT correspondence. To derive the  $c$ -function, we calculate the conformal anomaly (CA) in SUGRA, since it is closely related to the central charge. In this thesis, we discuss the various aspects of CA from AdS/CFT duality, especially for the cases of SUGRA in 3 and 5-dimensions which correspond to 2 and 4-dimensional CFTs, respectively. It is known that the bosonic part of SUGRA with scalar (dilaton) and arbitrary scalar potential describes the special RG flows in dual quantum field theory. So we calculate dilaton-dependent CA from dilatonic gravity with arbitrary potential. After that, we propose candidates of  $c$ -functions from such dilatonic gravity and investigate the properties of them.

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\*This paper is a part of the doctor thesis submitted to Graduate School of Humanities and Sciences, Ochanomizu University on December 2000, which is based on the works with S. Nojiri (National Defence Academy) and S.D. Odintsov (Universidad de Guanajuato).

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# 1 Introduction

In the last few years, there were a lot of attention related to AdS/CFT duality [1]. It is a duality between the large  $N$  limit of Conformal Field Theories (CFT) in  $d$ -dimensions and Supergravity (SUGRA) in  $d + 1$ -dimensional Anti de Sitter (AdS) space. By using this conjecture, we can study several properties of large  $N$  CFT by simple calculations in SUGRA side [2, 3]. The main idea of this proposal is that  $d + 1$ -dimensional SUGRA on AdS should be supplemented by the fields on the  $d$ -dimensional boundary of AdS, which is a CFT in  $d$ -dimensions. As the most interesting example, we can see the duality between  $\mathcal{N} = 4, SU(N)$  SYM theory in 4-dimensions and the type IIB string theory compactified on  $AdS_5 \times S^5$ .

Recently there are much attention for studying of Renormalization Group (RG) flow from SUGRA side [4, 5, 6, 7, 8, 9, 10, 11, 12] (and refs. therein). To describe c-function from AdS/CFT correspondence, we review briefly a general discussion of deformations in field theory and its dual description based on the report [13]. The deformation of CFT are made by adding the terms which break conformal invariance but keep Lorentz invariance,

$$S_{\text{CFT}} \rightarrow S_{\text{CFT}} + \int d^d x \phi \Phi(\phi).$$

$\Phi(\phi)$  is the local operator whose conformal dimension is  $\Delta$  and the coefficient  $\phi$  which can be regarded as a coupling constant in CFT has  $d - \Delta$  dimension. But it is the field in SUGRA in AdS background. The running of the coupling constant represents RG flow in CFT side, and this corresponds to the radial coordinate dependence of the field in AdS. Near the boundary of AdS, the field  $\phi$  behaves as

$$\phi(x, U) \xrightarrow{U \rightarrow \infty} U^{\Delta-d} \phi_{(0)}(x),$$

where  $\phi_{(0)}$  is the boundary value of AdS background. If there are scalar mass terms in AdS side, the classical equation of motion leads to the relation between conformal dimension  $\Delta$  and scalar mass  $m$ , the radius  $l$  of AdS as

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + l^2 m^2}.$$

If SUGRA theory has only massless scalar, the deformation of CFT is marginal which does not break conformal invariance, but other cases including mass terms in AdS side correspond to the relevant or irrelevant deformations in CFT. So the mass term is important here. Where do the mass terms come from? The terms come from the scalar potential terms in SUGRA.

In general, the scalar potential in SUGRA side has a very complicated form (the construction of 5-dimensional gauged SUGRA is given in ref. [14, 15]). So then we need to expand the potential around the stationary points which is given by the variation of the potential with respect to scalar  $\phi$ . Thus we can obtain mass terms of the scalar, and this scalar has the dependence of radial coordinate  $U$ . On the CFT side,  $\phi$  is the coupling constant which has fixed points given by the variation of  $\phi$  with respect to energy scale  $U$ . The scale  $U$  corresponds to the radial coordinate on the AdS side. On this fixed point,

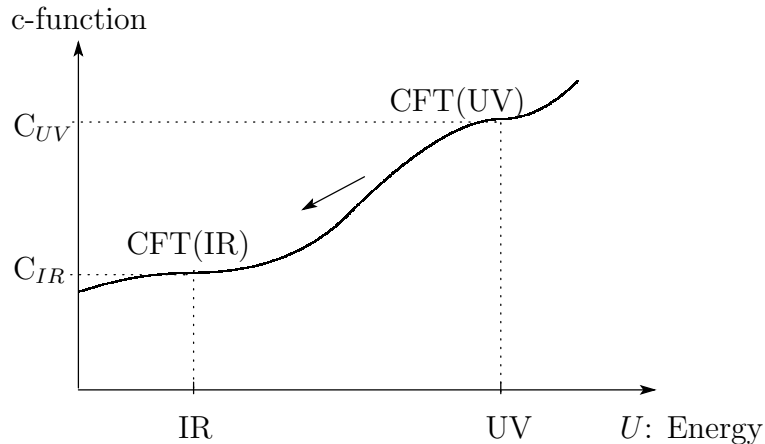


Figure 1: The behavior of c-function

the theory is conformal field theory having central charge  $c$ . The RG flow represents the changing of the central charges. In order to understand the relationship between different conformal field theories in 2-dimensions, Zamolodchikov's c-function is a great tool [16]. The properties of c-function are known as c-theorem given by Zamolodchikov as follows. First, this function is positive. Second, c-function is monotonically increasing function of the energy scale. This represents that RG transformation leads to a loss of information about the short distance degrees of freedom in the theory. Third, the function has fixed points which agree to the central charges. The behavior of c-function is summarized in Fig.1. The extensions of above c-theorem (c-function) to 4-dimensional field theory have been proposed by Cardy [17].

On the AdS side, RG flow corresponds to the geometrical changes (shown in Fig.2). For examples, the compact manifold  $S^5$  has  $SO(6)$  symmetry which represents R-symmetry of  $\mathcal{N} = 4$  in 4-dimensional CFT. The changing of the compact manifold from  $S^5$  to some  $S^5/M$ , ( $M$ : compact manifold) corresponds to the changing of SUSY number of CFT.

Now let us move on to the problem how to define c-function from AdS/CFT. It is well known in 2-dimensions that the central charge can be determined by the anomaly calculation as

$$\langle T^\mu_\mu \rangle = \frac{c}{24\pi} R.$$

The coefficient of the scalar curvature  $R$  corresponds to the central charge.

What is c-function in AdS side? Getting the idea from the relation between the central charge and the anomaly, we calculated the Conformal Anomaly (CA) by using the method which based on the work [18]. This method is focused on the conformal invariance of  $d+1$ -dimensional AdS gravity action, the breaking of this invariance corresponds to the CA in  $d$ -dimensional CFT<sup>1</sup>. It is possible to extend this method including scalar fields [24, 25]. The dilaton dependent holographic CA<sup>2</sup> has its counterpart as the same way in usual QFT CA for dilaton coupled theories [28, 29, 30].

<sup>1</sup>In this sense, this CA doesn't mean CA for SUGRA side. CA for SUGRA is also important for quantum cosmology [19, 20, 21]. For quantum gravity theory, see, for examples, [22, 23].

<sup>2</sup>From the point of brane-world scenario, CA was discussed in [26, 27].

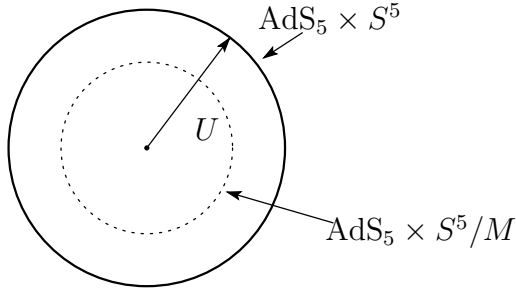


Figure 2: Geometrical changes in  $\text{AdS}_5 \times S^5$

The next section is devoted to the evaluation of CA from gauged SUGRA with arbitrary dilatonic potential via AdS/CFT correspondence. We present explicit result for 3 and 5-dimensional gauged SUGRAs. Such SUGRA side CA should correspond to dual QFT with broken conformal invariance in 2 and 4-dimensions, respectively. The explicit form of 4-dimensional CA takes few pages, so its lengthy dilaton-dependent coefficients are listed in Appendix A. The comparison with similar AdS/CFT calculation of CA in the same theory but with constant dilatonic potential is given. The candidates for c-function in 2 and 4-dimensions are proposed in section 3. Having examined some examples of scalar potentials, we checked the c-theorem and compared this c-function with the other proposals for it. Those two sections based on the works [31, 32].

From another side, the fundamental holographic principle [33] in AdS/CFT form enriches the classical gravity itself (and here also classical gauged SUGRA). Indeed, instead of the standard subtraction of reference background [34, 35] in making the gravitational action finite and the quasilocal stress tensor well-defined one introduces more elegant, local surface counterterm prescription [36]. Within it one adds the coordinate invariant functional of the intrinsic boundary geometry to gravitational action. Clearly, that does not modify the equations of motion. Moreover, this procedure has nice interpretation in terms of dual QFT as standard regularization. The specific choice of surface counterterm cancels the divergences of bulk gravitational action. As a by-product, it also defines the CA of boundary QFT.

Local surface counterterm prescription has been successfully applied to construction of finite action and quasilocal stress tensor on asymptotically AdS space in Einstein gravity [36, 37, 38, 39, 40]<sup>3</sup> and in higher derivative gravity [41]. Moreover, the generalization to asymptotically flat spaces is possible as it was first mentioned in ref.[42]. Surface counterterm has been found for domain-wall black holes in gauged SUGRA in diverse dimensions [43]. However, actually only the case of asymptotically constant dilaton has been investigated there.

In section 4, we construct such surface counterterms for 3 and 5-dimensional gauged SUGRAs based on the work [32]. As a result, the gravitational action in asymptotically AdS space is finite. On the same time, the gravitational stress tensor around such space

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<sup>3</sup>The method in [39] is not exactly counterterm method but based on the Noether theorem for diffeomorphism symmetry.

is well defined. It is interesting that CA defined in section 2 directly follows from the gravitational stress tensor with account of surface terms.

Section 5 is devoted to the application of finite gravitational action found in section 4 in the calculation of thermodynamical quantities of dilatonic AdS black hole. The dilatonic AdS black hole is constructed approximately, using the perturbations around constant dilaton AdS black hole. The entropy, mass and free energy of such black hole are found using the local surface counterterm prescription to regularize these quantities. The comparison is done with the case when standard prescription: regularization with reference background is used. The explicit regularization dependence of the result is mentioned.

The classical AdS-like solutions of 5-dimensional gauged SUGRA after the expansion over radial coordinate may be also used to get holographic CA for dual QFT as mentioned above. The calculation of holographic CA in such scheme gives very useful check of AdS/CFT correspondence, especially for Yang-Mills theory with maximally SUSY. Bosonic sector of 3 and 5-dimensional gauged SUGRA with specific parametrization of full scalar coset is considered (multi-dilaton gravity). In section 6, we discuss 3 and 5-dimensional gauged SUGRA with maximally SUSY based on [44], which is the extension of the previous works [31, 32] to multi-scalars case.

In section 7, we consider the scheme dependence of CA calculations from AdS/CFT correspondence. This section based on the work [45]. Usually, multi-loop quantum calculation is almost impossible to do, the result is known only in couple first orders of loop expansion, hence use of holographic CA is a challenge. CA for interacting QFT may be expressed in terms of gravitational invariants multiplied to multi-loop QFT beta-functions (see ref.[46] for recent discussion). One of the features of multi-loop beta-functions for coupling constants is their explicit scheme dependence (or regularization dependence) which normally occurs beyond second loop. This indicates that making calculation of holographic CA which corresponds to dual interacting QFT in different schemes leads also to scheme dependence of such CA. Of course, this should happen in the presence of non-trivial dilaton(s) and bulk potential.

Recently, in refs.[47] (see also [48, 49]) there appeared formulation of holographic RG based on Hamilton-Jacobi approach. This formalism permits to find the holographic CA without using the expansion of metric and dilaton over radial coordinate in AdS-like space. The purpose of this study is to calculate holographic CA for multi-dilaton gravity with non-trivial bulk potential in Boer-Verlinde-Verlinde formalism [47]. Then, the coefficients of curvature invariants as functions of bulk potential are obtained. The comparison of these coefficients (c-functions) with the ones found earlier [31, 32, 44] in the scheme of ref.[18] is done. It shows that coefficients coincide only when bulk potential is constant, in other words, holographic CA including non-constant bulk potential is scheme dependent.

In section 8, we studied holographic CA in higher dimensions by using Hamilton-Jacobi formalism. Especially we calculated 8-dimensional holographic CA and consider  $\text{AdS}_9/\text{CFT}_8$  correspondence [50]. Since the calculation of them is very complicate, we summarize the detailed calculations in Appendix E.

In this thesis, we discussed the various aspects of CA via AdS/CFT duality, especially for bosonic sector of 3 and 5-dimensional gauged SUGRA including single or multi-scalar with potential terms. Then we proposed c-function for the most simple case.

In the last section, we summarize the results and mention some open problems.

## 2 Conformal Anomaly for Gauged Supergravity with General Dilaton Potential

In the present section the derivation of dilaton-dependent Conformal Anomaly (CA) from gauged Supergravity (SUGRA) will be given.<sup>4</sup> As we will note in section 4 this derivation can be made also from the definition of finite action in asymptotically AdS space.

We start from the bulk action of  $d + 1$ -dimensional dilatonic gravity with arbitrary potential  $\Phi$

$$S = \frac{1}{16\pi G} \int_{M_{d+1}} d^{d+1}x \sqrt{-\hat{G}} \left\{ \hat{R} + X(\phi)(\hat{\nabla}\phi)^2 + Y(\phi)\hat{\Delta}\phi + \Phi(\phi) + 4\lambda^2 \right\} . \quad (2.1)$$

Here  $M_{d+1}$  is  $d + 1$ -dimensional manifold whose boundary is  $d$ -dimensional manifold  $M_d$  and we choose  $\Phi(0) = 0$ . Such action corresponds to (bosonic sector) of gauged SUGRA with single scalar (special RG flow). In other words, one considers RG flow in extended SUGRA when scalars lie in 1-dimensional submanifold of complete scalars space. Note also that classical vacuum stability restricts the form of dilaton potential [51]. As well-known, we also need to add the surface terms [34] to the bulk action in order to have well-defined variational principle. At the moment, for the purpose of calculation of CA (via AdS/CFT correspondence) the surface terms are irrelevant. The equations of motion given by variation of (2.1) with respect to  $\phi$  and  $G^{\mu\nu}$  are

$$0 = -\sqrt{-\hat{G}}\hat{\Phi}'(\phi) - \sqrt{-\hat{G}}V'(\phi)\hat{G}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 2\partial_\mu \left( \sqrt{-\hat{G}}\hat{G}^{\mu\nu}V(\phi)\partial_\nu\phi \right) , \quad (2.2)$$

$$0 = \frac{1}{d-1}\hat{G}_{\mu\nu} \left( \Phi(\phi) + \frac{d(d-1)}{l^2} \right) + \hat{R}_{\mu\nu} + V(\phi)\partial_\mu\phi\partial_\nu\phi . \quad (2.3)$$

Here

$$V(\phi) \equiv X(\phi) - Y'(\phi) . \quad (2.4)$$

where  $'$  denotes the derivative with respect to  $\phi$ . We choose the metric  $\hat{G}_{\mu\nu}$  on  $M_{d+1}$  and the metric  $\hat{g}_{\mu\nu}$  on  $M_d$  in the following form

$$ds^2 \equiv \hat{G}_{\mu\nu}dx^\mu dx^\nu = \frac{l^2}{4}\rho^{-2}d\rho d\rho + \sum_{i=1}^d \hat{g}_{ij}dx^i dx^j , \quad \hat{g}_{ij} = \rho^{-1}g_{ij} . \quad (2.5)$$

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<sup>4</sup>The conventions for the calculations are as follows;

$$\begin{aligned} R^\mu_{\nu\lambda\sigma} &\equiv \partial_\lambda \Gamma^\mu_{\sigma\nu} + \Gamma^\mu_{\lambda\rho} \Gamma^\rho_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\lambda\nu} - \Gamma^\mu_{\sigma\rho} \Gamma^\rho_{\lambda\nu} , \\ R_{\mu\nu} &\equiv R^\rho_{\mu\rho\nu}, \quad R \equiv G^{\mu\nu} R_{\mu\nu} , \\ \Gamma^\kappa_{\mu\nu} &= \frac{1}{2} G^{\kappa\rho} (G_{\mu\rho,\nu} + G_{\nu\rho,\mu} - G_{\mu\nu,\rho}) . \end{aligned}$$

Here  $l$  is related with  $\lambda^2$  by  $4\lambda^2 = d(d-1)/l^2$ . If  $g_{ij} = \eta_{ij}$ , the boundary of AdS lies at  $\rho = 0$ . We follow to method of calculation of CA as it was done in refs.[24, 25] where dilatonic gravity with constant dilaton potential has been considered.

The action (2.1) diverges in general since it contains the infinite volume integration on  $M_{d+1}$ . The action is regularized by introducing the infrared cutoff  $\epsilon$  and replacing

$$\int d^{d+1}x \rightarrow \int d^d x \int_{\epsilon} d\rho, \quad \int_{M_d} d^d x (\dots) \rightarrow \int d^d x (\dots) \Big|_{\rho=\epsilon}. \quad (2.6)$$

We also expand  $g_{ij}$  and  $\phi$  with respect to  $\rho$ :

$$g_{ij} = g_{(0)ij} + \rho g_{(1)ij} + \rho^2 g_{(2)ij} + \dots, \quad \phi = \phi_{(0)} + \rho \phi_{(1)} + \rho^2 \phi_{(2)} + \dots. \quad (2.7)$$

Then the action is also expanded as a power series on  $\rho$ . The subtraction of the terms proportional to the inverse power of  $\epsilon$  does not break the invariance under the scale transformation  $\delta g_{\mu\nu} = 2\delta\sigma g_{\mu\nu}$  and  $\delta\epsilon = 2\delta\sigma\epsilon$ . When  $d$  is even, however, the term proportional to  $\ln\epsilon$  appears. This term is not invariant under the scale transformation and the subtraction of the  $\ln\epsilon$  term breaks the invariance. The variation of the  $\ln\epsilon$  term under the scale transformation is finite when  $\epsilon \rightarrow 0$  and should be cancelled by the variation of the finite term (which does not depend on  $\epsilon$ ) in the action since the original action (2.1) is invariant under the scale transformation. Therefore the  $\ln\epsilon$  term  $S_{\ln}$  gives the CA  $T$  of the action renormalized by the subtraction of the terms which diverge when  $\epsilon \rightarrow 0$  ( $d = 4$ )

$$S_{\ln} = -\frac{1}{2} \int d^4 x \sqrt{-g} T. \quad (2.8)$$

The CA can be also obtained from the surface counterterms, which is discussed later in section 4.

First we consider the case of  $d = 2$ , i.e. 3-dimensional gauged SUGRA. The anomaly term  $S_{\ln}$  proportional to  $\ln\epsilon$  in the action is

$$S_{\ln} = -\frac{1}{16\pi G} \frac{l}{2} \int d^2 x \sqrt{-g_{(0)}} \left\{ R_{(0)} + X(\phi_{(0)}) (\nabla\phi_{(0)})^2 + Y(\phi_{(0)}) \Delta\phi_{(0)} \right. \\ \left. + \phi_{(1)} \Phi'(\phi_{(0)}) + \frac{1}{2} g_{(0)}^{ij} g_{(1)ij} \Phi(\phi_{(0)}) \right\}. \quad (2.9)$$

The terms proportional to  $\rho^0$  with  $\mu, \nu = i, j$  in (2.3) lead to  $g_{(1)ij}$  in terms of  $g_{(0)ij}$  and  $\phi_{(1)}$ .

$$g_{(1)ij} = \left[ -R_{(0)ij} - V(\phi_{(0)}) \partial_i \phi_{(0)} \partial_j \phi_{(0)} - g_{(0)ij} \Phi'(\phi_{(0)}) \phi_{(1)} \right. \\ \left. + \frac{g_{(0)ij}}{l^2} \left\{ 2\Phi'(\phi_{(0)}) \phi_{(1)} + R_{(0)} + V(\phi_{(0)}) g_{(0)}^{kl} \partial_k \phi_{(0)} \partial_l \phi_{(0)} \right\} \right. \\ \left. \times \left( \Phi(\phi_{(0)}) + \frac{2}{l^2} \right)^{-1} \right] \times \Phi(\phi_{(0)})^{-1} \quad (2.10)$$

In the equation (2.2), the terms proportional to  $\rho^{-1}$  lead to  $\phi_{(1)}$  as following.

$$\phi_{(1)} = \left[ V'(\phi_{(0)}) g_{(0)}^{ij} \partial_i \phi_{(0)} \partial_j \phi_{(0)} + 2 \frac{V(\phi_{(0)})}{\sqrt{-g_{(0)}}} \partial_i \left( \sqrt{-g_{(0)}} g_{(0)}^{ij} \partial_j \phi_{(0)} \right) \right]$$



$$\begin{aligned}
& + \frac{1}{2} \Phi'(\phi_{(0)}) \left( \Phi(\phi_{(0)}) + \frac{2}{l^2} \right)^{-1} \{ R_{(0)} + V(\phi_{(0)}) g_{(0)}^{ij} \partial_i \phi_{(0)} \partial_j \phi_{(0)} \} \Big] \\
& \times \left( \Phi''(\phi_{(0)}) - \Phi'(\phi_{(0)})^2 \left( \Phi(\phi_{(0)}) + \frac{2}{l^2} \right)^{-1} \right)^{-1} \quad (2.11)
\end{aligned}$$

Then anomaly term takes the following form using (2.10), (2.11)

$$\begin{aligned}
T = & \frac{1}{8\pi G} \frac{l}{2} \left\{ R_{(0)} + X(\phi_{(0)}) (\nabla \phi_{(0)})^2 + Y(\phi_{(0)}) \Delta \phi_{(0)} \right. \\
& + \frac{1}{2} \left\{ \frac{2\Phi'(\phi_{(0)})}{l^2} \left( \Phi''(\phi_{(0)}) \left( \Phi(\phi_{(0)}) + \frac{2}{l^2} \right) - \Phi'(\phi_{(0)})^2 \right)^{-1} - \Phi(\phi_{(0)}) \right\} \\
& \times \left( R_{(0)} + V(\phi_{(0)}) g_{(0)}^{ij} \partial_i \phi_{(0)} \partial_j \phi_{(0)} \right) \left( \Phi(\phi_{(0)}) + \frac{2}{l^2} \right)^{-1} \\
& + \frac{2\Phi'(\phi_{(0)})}{l^2} \left( \Phi''(\phi_{(0)}) \left( \Phi(\phi_{(0)}) + \frac{2}{l^2} \right) - \Phi'(\phi_{(0)})^2 \right)^{-1} \\
& \times \left( V'(\phi_{(0)}) g_{(0)}^{ij} \partial_i \phi_{(0)} \partial_j \phi_{(0)} + 2 \frac{V(\phi_{(0)})}{\sqrt{-g_{(0)}}} \partial_i \left( \sqrt{-g_{(0)}} g_{(0)}^{ij} \partial_j \phi_{(0)} \right) \right) \Big\} . \quad (2.12)
\end{aligned}$$

For  $\Phi(\phi) = 0$  case, the central charge of 2-dimensional conformal field theory is defined by the coefficient of  $R$ . Then it might be natural to introduce the candidate c-function  $c$  for the case when the conformal symmetry is broken by the deformation in the following way :

$$\begin{aligned}
c = & \frac{1}{2G} \left[ l + \frac{l}{2} \left\{ \frac{2\Phi'(\phi_{(0)})}{l^2} \left( \Phi''(\phi_{(0)}) \left( \Phi(\phi_{(0)}) + \frac{2}{l^2} \right) - \Phi'(\phi_{(0)})^2 \right)^{-1} - \Phi(\phi_{(0)}) \right\} \times \left( \Phi(\phi_{(0)}) + \frac{2}{l^2} \right)^{-1} \right] . \quad (2.13)
\end{aligned}$$

Comparing this with radiatively-corrected c-function of boundary QFT ( AdS<sub>3</sub>/CFT<sub>2</sub>) may help in correct bulk description of such theory. Clearly, that in the regions (or for potentials) where such candidate c-function is singular or not monotonic it cannot be the acceptable c-function. Presumably, the appearance of such regions indicates to the breaking of SUGRA description.

4-dimensional case is more interesting but also much more involved. The anomaly terms which proportional to  $\ln \epsilon$  are

$$\begin{aligned}
S_{\ln} = & \frac{1}{16\pi G} \int d^4x \sqrt{-g_{(0)}} \left[ \frac{-1}{2l} g_{(0)}^{ij} g_{(0)}^{kl} \left( g_{(1)ij} g_{(1)kl} - g_{(1)ik} g_{(1)jl} \right) \right. \\
& + \frac{l}{2} \left( R_{(0)}^{ij} - \frac{1}{2} g_{(0)}^{ij} R_{(0)} \right) g_{(1)ij} \\
& - \frac{2}{l} V(\phi_{(0)}) \phi_{(1)}^2 + \frac{l}{2} V'(\phi_{(0)}) \phi_{(1)} g_{(0)}^{ij} \partial_i \phi_{(0)} \partial_j \phi_{(0)} \\
& + l V(\phi_{(0)}) \phi_{(1)} \frac{1}{\sqrt{-g_{(0)}}} \partial_i \left( \sqrt{-g_{(0)}} g_{(0)}^{ij} \partial_j \phi_{(0)} \right) \\
& \left. + \frac{l}{2} V(\phi_{(0)}) \left( g_{(0)}^{ik} g_{(0)}^{jl} g_{(1)kl} - \frac{1}{2} g_{(0)}^{kl} g_{(1)kl} g_{(0)}^{ij} \right) \partial_i \phi_{(0)} \partial_j \phi_{(0)} \right] \quad (2.14)
\end{aligned}$$

$$\begin{aligned}
& -\frac{l}{2} \left( \frac{1}{2} g_{(0)}^{ij} g_{(2)ij} - \frac{1}{4} g_{(0)}^{ij} g_{(0)}^{kl} g_{(1)ik} g_{(1)jl} + \frac{1}{8} (g_{(0)}^{ij} g_{(1)ij})^2 \right) \Phi(\phi_{(0)}) \\
& -\frac{l}{2} \left( \Phi'(\phi_{(0)}) \phi_{(2)} + \frac{1}{2} \Phi''(\phi_{(0)}) \phi_{(1)}^2 + \frac{1}{2} g_{(0)}^{kl} g_{(1)kl} \Phi'(\phi_{(0)}) \phi_{(1)} \right) \Big] .
\end{aligned}$$

The terms proportional to  $\rho^0$  with  $\mu, \nu = i, j$  in the equation of the motion (2.3) lead to  $g_{(1)ij}$  in terms of  $g_{(0)ij}$  and  $\phi_{(1)}$ .

$$\begin{aligned}
g_{(1)ij} = & \left[ -R_{(0)ij} - V(\phi_{(0)}) \partial_i \phi_{(0)} \partial_j \phi_{(0)} - \frac{1}{3} g_{(0)ij} \Phi'(\phi_{(0)}) \phi_{(1)} \right. \\
& + \frac{g_{(0)ij}}{l^2} \left\{ \frac{4}{3} \Phi'(\phi_{(0)}) \phi_{(1)} + R_{(0)} + V(\phi_{(0)}) g_{(0)}^{kl} \partial_k \phi_{(0)} \partial_l \phi_{(0)} \right\} \\
& \left. \times \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{6}{l^2} \right)^{-1} \right] \times \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{2}{l^2} \right)^{-1} . \quad (2.15)
\end{aligned}$$

In the equation (2.2), the terms proportional to  $\rho^{-2}$  lead to  $\phi_{(1)}$  as follows:

$$\begin{aligned}
\phi_{(1)} = & \left[ V'(\phi_{(0)}) g_{(0)}^{ij} \partial_i \phi_{(0)} \partial_j \phi_{(0)} + 2 \frac{V(\phi_{(0)})}{\sqrt{-g_{(0)}}} \partial_i \left( \sqrt{-g_{(0)}} g_{(0)}^{ij} \partial_j \phi_{(0)} \right) \right. \\
& + \frac{1}{2} \Phi'(\phi_{(0)}) \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{6}{l^2} \right)^{-1} \left\{ R_{(0)} + V(\phi_{(0)}) g_{(0)}^{ij} \partial_i \phi_{(0)} \partial_j \phi_{(0)} \right\} \Big] \\
& \times \left( \frac{8V(\phi_{(0)})}{l^2} + \Phi''(\phi_{(0)}) - \frac{2}{3} \Phi'(\phi_{(0)})^2 \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{6}{l^2} \right)^{-1} \right)^{-1} . \quad (2.16)
\end{aligned}$$

In the equation (2.3), the terms proportional to  $\rho^1$  with  $\mu, \nu = i, j$  lead to  $g_{(2)ij}$ .

$$\begin{aligned}
g_{(2)ij} = & \left[ -\frac{1}{3} \left\{ g_{(1)ij} \Phi'(\phi_{(0)}) \phi_{(1)} + g_{(0)ij} (\Phi'(\phi_{(0)}) \phi_{(2)} + \frac{1}{2} \Phi''(\phi_{(0)}) \phi_{(1)}^2) \right\} \right. \\
& - \frac{2}{l^2} g_{(0)}^{kl} g_{(1)ki} g_{(1)lj} + \frac{1}{l^2} g_{(0)}^{km} g_{(0)}^{nl} g_{(1)mn} g_{(1)kl} g_{(0)ij} \\
& - \frac{2}{l^2} g_{(0)ij} \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{8}{l^2} \right)^{-1} \times \left\{ \frac{2}{l^2} g_{(0)}^{mn} g_{(0)}^{kl} g_{(1)km} g_{(1)ln} \right. \\
& - \frac{4}{3} \left( \Phi'(\phi_{(0)}) \phi_{(2)} + \frac{1}{2} \Phi''(\phi_{(0)}) \phi_{(1)}^2 \right) - \frac{1}{3} g_{(0)}^{ij} g_{(1)ij} \Phi'(\phi_{(0)}) \phi_{(1)} \\
& + V'(\phi_{(0)}) \phi_{(1)} g_{(0)}^{ij} \partial_i \phi_{(0)} \partial_j \phi_{(0)} + \frac{2V(\phi_{(0)}) \phi_{(1)}}{\sqrt{-g_{(0)}}} \partial_i \left( \sqrt{-g_{(0)}} g_{(0)}^{ij} \partial_j \phi_{(0)} \right) \Big\} \\
& + V'(\phi_{(0)}) \phi_{(1)} \partial_i \phi_{(0)} \partial_j \phi_{(0)} + 2V(\phi_{(0)}) \phi_{(1)} \partial_i \partial_j \phi_{(0)} \Big] \\
& \times \left( \frac{1}{3} \Phi(\phi_{(0)}) \right)^{-1} . \quad (2.17)
\end{aligned}$$

And the terms proportional to  $\rho^{-1}$  in the equation (2.2), lead to  $\phi_{(2)}$  as follows:

$$\begin{aligned}
\phi_{(2)} = & \left[ V''(\phi_{(0)}) \phi_{(1)} g_{(0)}^{ij} \partial_i \phi_{(0)} \partial_j \phi_{(0)} \right. \\
& + V'(\phi_{(0)}) \left( g_{(0)}^{ik} g_{(0)}^{jl} - \frac{1}{2} g_{(0)}^{ij} g_{(0)}^{kl} \right) g_{(1)kl} \partial_i \phi_{(0)} \partial_j \phi_{(0)} \Big]
\end{aligned}$$

$$\begin{aligned}
& + \frac{2V'(\phi_{(0)})\phi_{(1)}}{\sqrt{-g_{(0)}}} \partial_i \left( \sqrt{-g_{(0)}} g_{(0)}^{ij} \partial_j \phi_{(0)} \right) \\
& - \frac{4}{l^2} V'_{(0)} \phi_{(1)}^2 - \frac{1}{2} \Phi'''(\phi_{(0)}) \phi_{(1)}^2 - \frac{1}{2} g_{(0)}^{kl} g_{(1)kl} \Phi''(\phi_{(0)}) \phi_{(1)} \\
& - \left( \frac{-1}{4} g_{(0)}^{ij} g_{(0)}^{kl} g_{(1)ik} g_{(1)jl} + \frac{1}{8} (g_{(0)}^{ij} g_{(1)ij})^2 \right) \Phi'(\phi_{(0)}) \\
& - \frac{1}{2} \Phi'(\phi_{(0)}) \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{8}{l^2} \right)^{-1} \times \left\{ \frac{2}{l^2} g_{(0)}^{mn} g_{(0)}^{kl} g_{(1)km} g_{(1)ln} \right. \\
& - \frac{2}{3} \Phi''(\phi_{(0)}) \phi_{(1)}^2 - \frac{1}{3} g_{(0)}^{ij} g_{(1)ij} \Phi'(\phi_{(0)}) \phi_{(1)} \\
& \left. + V'(\phi_{(0)}) \phi_{(1)} g_{(0)}^{ij} \partial_i \phi_{(0)} \partial_j \phi_{(0)} + \frac{2V(\phi_{(0)})\phi_{(1)}}{\sqrt{-g_{(0)}}} \partial_i \left( \sqrt{-g_{(0)}} g_{(0)}^{ij} \partial_j \phi_{(0)} \right) \right\} \\
& \times \left( \Phi''(\phi_{(0)}) - \frac{2}{3} \Phi'(\phi_{(0)})^2 \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{8}{l^2} \right)^{-1} \right)^{-1} \tag{2.18}
\end{aligned}$$

Then we can get the anomaly (2.14) in terms of  $g_{(0)ij}$  and  $\phi_{(0)}$ , which are boundary values of metric and dilaton respectively by using (2.15), (2.16), (2.17), (2.18). In the following, we choose  $l = 1$ , denote  $\Phi(\phi_{(0)})$  by  $\Phi$  and abbreviate the index (0) for the simplicity. Then substituting (2.16) into (2.15), we obtain

$$\begin{aligned}
g_{(1)ij} &= \tilde{c}_1 R_{ij} + \tilde{c}_2 g_{ij} R + \tilde{c}_3 g_{ij} g^{kl} \partial_k \phi \partial_l \phi \\
&+ \tilde{c}_4 g_{ij} \frac{\partial_k}{\sqrt{-g}} \left( \sqrt{-g} g^{kl} \partial_l \phi \right) + \tilde{c}_5 \partial_i \phi \partial_j \phi . \tag{2.19}
\end{aligned}$$

The explicit form of  $\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_5$  is given in Appendix A. Further, substituting (2.16) and (2.19) into (2.18), one gets

$$\begin{aligned}
\phi_{(2)} &= d_1 R^2 + d_2 R_{ij} R^{ij} + d_3 R^{ij} \partial_i \phi \partial_j \phi \\
&+ d_4 R g^{ij} \partial_i \phi \partial_j \phi + d_5 R \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \\
&+ d_6 (g^{ij} \partial_i \phi \partial_j \phi)^2 + d_7 \left( \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right)^2 \\
&+ d_8 g^{kl} \partial_k \phi \partial_l \phi \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) . \tag{2.20}
\end{aligned}$$

Here, the explicit form of  $d_1, \dots, d_8$  is given in Appendix A. Substituting (2.16), (2.19) and (2.20) into (2.17), one gets

$$\begin{aligned}
g^{ij} g_{(2)ij} &= f_1 R^2 + f_2 R_{ij} R^{ij} + f_3 R^{ij} \partial_i \phi \partial_j \phi \\
&+ f_4 R g^{ij} \partial_i \phi \partial_j \phi + f_5 R \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \\
&+ f_6 (g^{ij} \partial_i \phi \partial_j \phi)^2 + f_7 \left( \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right)^2 \\
&+ f_8 g^{kl} \partial_k \phi \partial_l \phi \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) . \tag{2.21}
\end{aligned}$$

Again, the explicit form of very complicated functions  $f_1, \dots, f_8$  is given in Appendix A. Finally substituting (2.16), (2.19), (2.20) and (2.21) into the expression for the anomaly (2.14), we obtain,

$$\begin{aligned}
T = & -\frac{1}{8\pi G} \left[ h_1 R^2 + h_2 R_{ij} R^{ij} + h_3 R^{ij} \partial_i \phi \partial_j \phi \right. \\
& + h_4 R g^{ij} \partial_i \phi \partial_j \phi + h_5 R \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \\
& + h_6 (g^{ij} \partial_i \phi \partial_j \phi)^2 + h_7 \left( \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right)^2 \\
& \left. + h_8 g^{kl} \partial_k \phi \partial_l \phi \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right] . \tag{2.22}
\end{aligned}$$

Here

$$\begin{aligned}
h_1 = & \left[ 3 \left\{ (24 - 10 \Phi) \Phi^6 \right. \right. \\
& + (62208 + 22464 \Phi + 2196 \Phi^2 + 72 \Phi^3 + \Phi^4) \Phi'' (\Phi'' + 8 V)^2 \\
& + 2 \Phi^4 \left\{ (108 + 162 \Phi + 7 \Phi^2) \Phi'' + 72 (-8 + 14 \Phi + \Phi^2) V \right\} \\
& - 2 \Phi^2 \left\{ (6912 + 2736 \Phi + 192 \Phi^2 + \Phi^3) \Phi''^2 \right. \\
& + 4 (11232 + 6156 \Phi + 552 \Phi^2 + 13 \Phi^3) \Phi'' V \\
& + 32 (-2592 + 468 \Phi + 96 \Phi^2 + 5 \Phi^3) V^2 \left. \right\} \\
& \left. - 3 (-24 + \Phi) (6 + \Phi)^2 \Phi'^3 (\Phi''' + 8 V') \right\} \Big] / \\
& \left[ 16 (6 + \Phi)^2 \left\{ -2 \Phi'^2 + (24 + \Phi) \Phi'' \right\} \left\{ -2 \Phi'^2 \right. \right. \\
& \left. \left. + (18 + \Phi) (\Phi'' + 8 V) \right\}^2 \right] \\
h_2 = & -\frac{3 \left\{ (12 - 5 \Phi) \Phi'^2 + (288 + 72 \Phi + \Phi^2) \Phi'' \right\}}{8 (6 + \Phi)^2 \left\{ -2 \Phi'^2 + (24 + \Phi) \Phi'' \right\}} . \tag{2.23}
\end{aligned}$$

We also give the explicit forms of  $h_3, \dots, h_8$  in Appendix A. Thus, we found the complete CA from bulk side. This expression which should describe dual 4-dimensional QFT of QCD type, with broken SUSY looks really complicated. The interesting remark is that CA is not integrable in general. In other words, it is impossible to construct the anomaly induced action. This is not strange, as it is usual situation for CA when radiative corrections are taken into account.

In case of the dilaton gravity in [24] corresponding to  $\Phi = 0$  (or more generally in case that the axion is included [52] as in [25]), we have the following expression:

$$\begin{aligned}
T = & \frac{l^3}{8\pi G} \int d^4 x \sqrt{-g_{(0)}} \left[ \frac{1}{8} R_{(0)ij} R_{(0)}^{ij} - \frac{1}{24} R_{(0)}^2 \right. \\
& - \frac{1}{2} R_{(0)}^{ij} \partial_i \varphi_{(0)} \partial_j \varphi_{(0)} + \frac{1}{6} R_{(0)} g_{(0)}^{ij} \partial_i \varphi_{(0)} \partial_j \varphi_{(0)} \\
& \left. + \frac{1}{4} \left\{ \frac{1}{\sqrt{-g_{(0)}}} \partial_i (\sqrt{-g_{(0)}} g_{(0)}^{ij} \partial_j \varphi_{(0)}) \right\}^2 + \frac{1}{3} (g_{(0)}^{ij} \partial_i \varphi_{(0)} \partial_j \varphi_{(0)})^2 \right] . \tag{2.24}
\end{aligned}$$

Here  $\varphi$  can be regarded as dilaton. In the limit of  $\Phi \rightarrow 0$ , we obtain

$$\begin{aligned}
h_1 &\rightarrow \frac{3 \cdot 62208 \Phi'' (8V)^2}{16 \cdot 6^2 \cdot 24 \cdot 18^2 \Phi'' (8V)^2} = \frac{1}{24} \\
h_2 &\rightarrow -\frac{3 \cdot 288 \Phi''}{8 \cdot 6^2 \cdot 24 \Phi''} = -\frac{1}{8} \\
h_3 &\rightarrow -\frac{3 \cdot 288 (\Phi'' V - \Phi' V')}{4 \cdot 6^2 \cdot 24 \Phi''} = -\frac{1}{4} \frac{(\Phi'' V - \Phi' V')}{\Phi''} \\
h_4 &\rightarrow \frac{3 \cdot 62208 \Phi'' V (8V)^2 + 6 \Phi' \cdot 384 \cdot (-5184) \cdot V^2 V'}{8 \cdot 6^2 \cdot 24 \Phi'' \cdot (18 \cdot 8V)^2} = \frac{1}{12} \frac{(\Phi'' V - \Phi' V')}{\Phi''} \\
h_5 &\rightarrow 0 \\
h_6 &\rightarrow \left\{ -\Phi'' \cdot 64V \cdot (373248V^3 - 139968V'^2) \right. \\
&\quad \left. + 2 \cdot 6 \Phi' V' \cdot (-2) \cdot (-432) \cdot (4608V^3 + 864V'^2 - 1728VV'') \right\} \\
&\quad / 16 \cdot 6^2 \cdot 24 \Phi'' \cdot (18 \cdot 8V)^2 \\
&= \frac{\left\{ -\Phi'' V \cdot (V^3 - \frac{3}{8} V'^2) + 2 \Phi' V' \cdot (V^3 + \frac{3}{16} V'^2 - \frac{3}{8} V V'') \right\}}{12 \Phi'' V^2} \\
h_7 &\rightarrow \frac{V \cdot 8 \cdot 18^2 \Phi'' V \cdot 2 \cdot 12V}{24 \Phi'' \cdot (18 \cdot 8V)^2} = \frac{V}{8} \\
h_8 &\rightarrow \frac{32 \cdot 18^2 \Phi'' V \cdot 2 \cdot 12 \cdot V'}{4 \cdot 24 \Phi'' (18 \cdot 8V)^2} = \frac{V'}{8V} .
\end{aligned} \tag{2.25}$$

Especially if we choose

$$V = -2 , \tag{2.26}$$

we obtain,

$$\begin{aligned}
h_1 &\rightarrow \frac{1}{24} , \quad h_2 \rightarrow -\frac{1}{8} , \quad h_3 \rightarrow \frac{1}{2} , \quad h_4 \rightarrow -\frac{1}{6} \\
h_5 &\rightarrow 0 , \quad h_6 \rightarrow -\frac{1}{3} , \quad h_7 \rightarrow -\frac{1}{4} , \quad h_8 \rightarrow 0
\end{aligned} \tag{2.27}$$

and we find that the standard result (CA of  $\mathcal{N} = 4$  super YM theory covariantly coupled with  $\mathcal{N} = 4$  conformal SUGRA [53, 54, 55]) in (2.24) is reproduced [24, 56].

We should also note that the expression (2.22) cannot be rewritten as a sum of the Gauss-Bonnet invariant  $G$  and the square of the conformal tensor  $F$ , which are given as

$$\begin{aligned}
G &= R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl} \\
F &= \frac{1}{3}R^2 - 2R_{ij}R^{ij} + R_{ijkl}R^{ijkl} ,
\end{aligned} \tag{2.28}$$

This is the signal that the conformal symmetry is broken already in classical theory.

When  $\phi$  is constant, only two terms corresponding to  $h_1$  and  $h_2$  survive in (2.22) :

$$\begin{aligned}
T &= -\frac{1}{8\pi G} [h_1 R^2 + h_2 R_{ij}R^{ij}] \\
&= -\frac{1}{8\pi G} \left[ \left( h_1 + \frac{1}{3} h_2 \right) R^2 + \frac{1}{2} h_2 (F - G) \right] .
\end{aligned} \tag{2.29}$$

As  $h_1$  depends on  $V$ , we may compare the result with the CA from, say, scalar or spinor QED, or QCD in the phase where there are no background scalars and (or) spinors. The structure of the CA in such a theory has the following form

$$T = \hat{a}G + \hat{b}F + \hat{c}R^2 . \quad (2.30)$$

where

$$\hat{a} = \text{constant} + a_1 e^2 , \quad \hat{b} = \text{constant} + a_2 e^2 , \quad \hat{c} = a_3 e^2 . \quad (2.31)$$

Here  $e^2$  is the electric charge (or  $g^2$  in case of QCD). Imagine that one can identify  $e$  with the exponential of the constant dilaton (using holographic RG [57, 47]).  $a_1$ ,  $a_2$  and  $a_3$  are some numbers. Comparing (2.29) and (2.30), we obtain

$$\hat{a} = -\hat{b} = \frac{h_2}{16\pi G} , \quad \hat{c} = -\frac{1}{8\pi G} \left( h_1 + \frac{1}{3}h_2 \right) . \quad (2.32)$$

When  $\Phi$  is small, one gets

$$\begin{aligned} h_1 &= \frac{1}{24} \left[ 1 - \frac{1}{8}\Phi + \frac{1}{8} \frac{(\Phi')^2}{\Phi''} \right. \\ &\quad \left. + \frac{25}{2592}\Phi^2 - \frac{17}{216} \frac{(\Phi')^2 \Phi}{\Phi''} + \frac{1}{576} \frac{(\Phi')^2}{V} + \frac{1}{96} \frac{(\Phi')^4}{(\Phi'')^2} + \mathcal{O}(\Phi^3) \right] \\ h_2 &= -\frac{1}{8} \left[ 1 - \frac{1}{8}\Phi + \frac{1}{8} \frac{(\Phi')^2}{\Phi''} \right. \\ &\quad \left. + \frac{5}{576}\Phi^2 - \frac{3}{64} \frac{(\Phi')^2 \Phi}{\Phi''} + \frac{1}{96} \frac{(\Phi')^4}{(\Phi'')^2} + \mathcal{O}(\Phi^3) \right] . \end{aligned} \quad (2.33)$$

If one assumes

$$\Phi(\phi) = a e^{b\phi} , \quad (|a| \ll 1) , \quad (2.34)$$

then

$$\begin{aligned} h_2 &= -\frac{1}{8} \left[ 1 - \frac{a^2}{36} e^{2b\phi} + \mathcal{O}(a^3) \right] \\ h_1 + \frac{1}{3}h_2 &= \frac{a^2}{24} \left( -\frac{5}{162} + \frac{b^2}{576V} \right) e^{2b\phi} + \mathcal{O}(a^3) . \end{aligned} \quad (2.35)$$

Comparing (2.35) with (2.31) and (2.32) and assuming

$$e^2 = e^{2b\phi} , \quad (2.36)$$

we find

$$\begin{aligned} a_1 = -a_2 &= \frac{1}{16\pi G} \cdot \frac{1}{8} \cdot \frac{a^2}{36} , \\ a_3 &= -\frac{1}{8\pi G} \cdot \frac{a^2}{24} \cdot \left( -\frac{5}{162} + \frac{b^2}{576V} \right) . \end{aligned} \quad (2.37)$$

Here  $V$  should be arbitrary but constant. We should note  $\Phi(0) \neq 0$ . One can absorb the difference into the redefinition of  $l$  since we need not to assume  $\Phi(0) = 0$  in deriving the form of  $h_1$  and  $h_2$  in (2.23). Hence, this simple example suggests the way of comparison between SUGRA side and QFT descriptions of non-conformal boundary theory.

In order that the region near the boundary at  $\rho = 0$  is asymptotically AdS, we need to require  $\Phi \rightarrow 0$  and  $\Phi' \rightarrow 0$  when  $\rho \rightarrow 0$ . One can also confirm that  $h_1 \rightarrow \frac{1}{24}$  and  $h_2 \rightarrow -\frac{1}{8}$  in the limit of  $\Phi \rightarrow 0$  and  $\Phi' \rightarrow 0$  even if  $\Phi'' \neq 0$  and  $\Phi''' \neq 0$ . In the AdS/CFT correspondence,  $h_1$  and  $h_2$  are related with the central charge  $c$  of the conformal field theory (or its analog for non-conformal theory). Since we have two functions  $h_1$  and  $h_2$ , there are two ways to define the candidate c-function when the conformal field theory is deformed:

$$c_1 = \frac{24\pi h_1}{G}, \quad c_2 = -\frac{8\pi h_2}{G}. \quad (2.38)$$

If we put  $\mathcal{V}(\phi) = 4\lambda^2 + \Phi(\phi)$ , then  $l = \left(\frac{12}{\mathcal{V}(0)}\right)^{\frac{1}{2}}$ . One should note that it is chosen  $l = 1$  in (2.38). We can restore  $l$  by changing  $h \rightarrow l^3 h$  and  $k \rightarrow l^3 k$  and  $\Phi' \rightarrow l\Phi'$ ,  $\Phi'' \rightarrow l^2\Phi''$  and  $\Phi''' \rightarrow l^3\Phi'''$  in (2.22). Then in the limit of  $\Phi \rightarrow 0$ , one gets

$$c_1, \quad c_2 \rightarrow \frac{\pi}{G} \left( \frac{12}{\mathcal{V}(0)} \right)^{\frac{3}{2}}, \quad (2.39)$$

which agrees with the proposal of the previous work [5] in the limit. The c-function  $c_1$  or  $c_2$  in (2.38) is, of course, more general definition. It is interesting to study the behavior of candidate c-function for explicit values of dilatonic potential at different limits. It also could be interesting to see what is the analogue of our dilaton-dependent c-function in non-commutative YM theory (without dilaton, see [58]).

### 3 Properties of c-function

The definitions of the c-functions in (2.13) and (2.38), are, however, not always good ones since the results are too wide. That is, we have obtained the CA for arbitrary dilatonic background which may not be the solution of original  $d = 5$  gauged SUGRA. As only solutions of 5-dimensional gauged SUGRA describe RG flows of dual QFT it is not strange that above candidate c-functions are not acceptable. They quickly become non-monotonic and even singular in explicit examples. They presumably measure the deviations from SUGRA description and should not be taken seriously. As pointed in [59], it might be necessary to impose the condition  $\Phi' = 0$  on the conformal boundary. Such condition follows from the equations of motion of 5-dimensional gauged SUGRA. Anyway as  $\Phi' = 0$  on the boundary in the solution which has the asymptotic AdS region, we can add any function which proportional to the power of  $\Phi' = 0$  to the previous expressions of the c-functions in (2.13) and (2.38). As a trial, if we put  $\Phi' = 0$ , we obtain

$$c = \frac{1}{2G} \left[ \frac{l}{2} + \frac{1}{l} \frac{1}{\Phi(\phi_{(0)}) + \frac{2}{l^2}} \right] \quad (3.1)$$

instead of (2.13) and

$$\begin{aligned} c_1 &= \frac{9\pi}{2G} \frac{62208 + 22464\Phi + 2196\Phi^2 + 72\Phi^3 + \Phi^4}{(6 + \Phi)^2(24 + \Phi)(18 + \Phi)^2} \\ c_2 &= \frac{3\pi}{G} \frac{288 + 72\Phi + \Phi^2}{(6 + \Phi)^2(24 + \Phi)} \end{aligned} \quad (3.2)$$

instead of (2.38). We should note that there disappear the higher derivative terms like  $\Phi''$  or  $\Phi'''$ . That will be our final proposal for acceptable c-function in terms of dilatonic potential. The given c-functions in (3.2) also have the property (2.39) and reproduce the known result for the central charge on the boundary. Since  $\frac{d\Phi}{dz} \rightarrow 0$  in the asymptotically AdS region even if the region is ultraviolet (UV) or infrared (IR), the given c-functions in (3.1) and (3.2) have fixed points in the asymptotic AdS region  $\frac{dc}{dU} = \frac{dc}{d\Phi} \frac{d\Phi}{d\phi} \frac{d\phi}{dU} \rightarrow 0$ , where  $U = \rho^{-\frac{1}{2}}$  is the radius coordinate in AdS or the energy scale of the boundary field theory.

We can now check the monotonicity of the c-functions. For this purpose, we consider some examples. In [4] and [5], the following dilaton potentials appeared:

$$4\lambda^2 + \Phi_{\text{FGPW}}(\phi) = 4 \left( \exp \left[ \left( \frac{4\phi}{\sqrt{6}} \right) \right] + 2 \exp \left[ - \left( \frac{2\phi}{\sqrt{6}} \right) \right] \right) \quad (3.3)$$

$$4\lambda^2 + \Phi_{\text{GPPZ}}(\phi) = \frac{3}{2} \left( 3 + \left( \cosh \left[ \left( \frac{2\phi}{\sqrt{3}} \right) \right] \right)^2 + 4 \cosh \left[ \left( \frac{2\phi}{\sqrt{3}} \right) \right] \right). \quad (3.4)$$

In both cases  $\mathcal{V}$  is a constant as  $\mathcal{V} = -2$ . In the classical solutions for the both cases,  $\phi$  is the monotonically decreasing function of the energy scale  $U = \rho^{-\frac{1}{2}}$  and  $\phi = 0$  at the UV limit corresponding to the boundary. Then in order to know the energy scale dependences of  $c_1$  and  $c_2$ , we only need to investigate the  $\phi$  dependences of  $c_1$  and  $c_2$  in (3.2). As the potentials and also  $\Phi$  have a minimum  $\Phi = 0$  at  $\phi = 0$ , which corresponds to the UV boundary in the solutions in [4] and [5], and  $\Phi$  is monotonically increasing function of the absolute value  $|\phi|$ , we only need to check the monotonicity of  $c_1$  and  $c_2$  with respect to  $\Phi$  when  $\Phi \geq 0$ . From (3.2), we find

$$\begin{aligned} &\frac{d(\ln c_1)}{d\Phi} \\ &= - \frac{20155392 + 12006144\Phi + 2209680\Phi^2 + 180576\Phi^3 + 6840\Phi^4 + 120\Phi^5 + \Phi^6}{(6 + \Phi)(18 + \Phi)(24 + \Phi)(62208 + 22464\Phi + 2196\Phi^2 + 72\Phi^3 + \Phi^4)} \\ &< 0 \\ &\frac{d(\ln c_2)}{d\Phi} = - \frac{5184 + 2304\Phi + 138\Phi^2 + \Phi^3}{(6 + \Phi)(24 + \Phi)(288 + 72\Phi + \Phi^2)} < 0. \end{aligned} \quad (3.5)$$

Therefore the c-functions  $c_1$  and  $c_2$  are monotonically decreasing functions of  $\Phi$  or increasing function of the energy scale  $U$  as the c-function in [6, 5]. We should also note that the c-functions  $c_1$  and  $c_2$  are positive definite for non-negative  $\Phi$ . For  $c$  in (3.1) for  $d = 2$  case, it is very straightforward to check the monotonicity and the positivity.

In [5], another c-function has been proposed in terms of the metric as follows:

$$c_{\text{GPPZ}} = \left( \frac{dA}{dz} \right)^{-3}, \quad (3.6)$$



where the metric is given by

$$ds^2 = dz^2 + e^{2A} dx_\mu dx^\mu . \quad (3.7)$$

The c-function (3.6) is positive and has a fixed point in the asymptotically AdS region again and the c-function is also monotonically increasing function of the energy scale. The c-functions (3.1) and (3.2) proposed in this thesis are given in terms of the dilaton potential, not in terms of metric, but it might be interesting that the c-functions in (3.1) and (3.2) have the similar properties (positivity, monotonically and fixed point in the asymptotically AdS region). These properties could be understood from the equations of motion. When the metric has the form (3.7), the equations of motion are:

$$\phi'' + dA'\phi' = \frac{\partial\Phi}{\partial\phi} , \quad (3.8)$$

$$dA'' + d(A')^2 + \frac{1}{2}(\phi')^2 = -\frac{4\lambda^2 + \Phi}{d-1} , \quad (3.9)$$

$$A'' + d(A')^2 = -\frac{4\lambda^2 + \Phi}{d-1} . \quad (3.10)$$

Here  $' \equiv \frac{d}{dz}$ . From (3.8) and (3.9), we obtain

$$0 = 2(d-1)A'' + \phi'^2 \quad (3.11)$$

If  $A'' = 0$ , then  $\phi' = 0$ , which tells that if we take  $\frac{dc_{\text{GPPZ}}}{dz} = 0$ , then  $\frac{dc_1}{dz} = \frac{dc_2}{dz} = 0$ . Thus if  $c_{\text{GPPZ}}$  has a fixed point,  $c_1$  and  $c_2$  also have a fixed point. From (3.8) and (3.9), we obtain

$$0 = d(d-1)A'^2 + 4\lambda^2 + \Phi - \frac{1}{2}\phi'^2 . \quad (3.12)$$

Then at the fixed point where  $\phi' = 0$ , we obtain

$$0 = d(d-1)A'^2 + 4\lambda^2 + \Phi . \quad (3.13)$$

Taking  $c_{\text{GPPZ}}$  and  $A'$  is the monotonic function of  $z$ , potential  $\mathcal{V}$  and  $c_1$  and  $c_2$  are also monotonic function at least at the fixed point. We have to note that above considerations do not give the proof of equivalency of our proposal c-functions with other proposals. However, it is remarkable (at least, for a number of potentials) that they enjoy the similar properties: positivity, monotonically and existence of fixed points.

We can also consider other examples of c-function for different choices of dilatonic potential. In [60, 61], several examples of the potentials in gauged SUGRA are given. They appeared as a result of sphere reduction in M-theory or string theory, down to 3 or 5-dimensions. Their properties are described in detail in refs.[60, 61]. The potentials have the following form:

$$4\lambda^2 + \Phi(\phi) = \frac{d(d-1)}{\frac{1}{a_1^2} - \frac{1}{a_1 a_2}} \left( \frac{1}{a_1^2} e^{a_1 \phi} - \frac{1}{a_1 a_2} e^{a_2 \phi} \right) . \quad (3.14)$$

Here  $a_1$  and  $a_2$  are constant parameters depending on the model. We also normalize the potential so that  $4\lambda^2 + \Phi(\phi) \rightarrow d(d-1)$  when  $\phi \rightarrow 0$ . For simplicity, we choose  $G = l = 1$  in this section.

For  $\mathcal{N} = 1$  model in  $D = d + 1 = 3$ -dimensions

$$a_1 = 2\sqrt{2} \ , \quad a_2 = \sqrt{2} \ , \quad (3.15)$$

for  $D = 3, \mathcal{N} = 2$ , one gets

$$a_1 = \sqrt{6} \ , \quad a_2 = 2\sqrt{\frac{2}{3}} \ , \quad (3.16)$$

and for  $D = 3, \mathcal{N} = 3$  model, we have

$$a_1 = \frac{4}{\sqrt{3}} \ , \quad a_2 = \sqrt{3} \ . \quad (3.17)$$

On the other hand, for  $D = d + 1 = 5, \mathcal{N} = 1$  model,  $a_1$  and  $a_2$  are

$$a_1 = 2\sqrt{\frac{5}{3}} \ , \quad a_2 = \frac{4}{\sqrt{15}} \ . \quad (3.18)$$

The proposed c-functions have not acceptable behavior for above potentials. (There seems to be no problem for 2-dimensional case.) The problem seems to be that the solutions in above models have not asymptotic AdS region in UV but in IR. On the same time the CA in (2.22) is evaluated as UV effect. If we assume that  $\Phi$  in the expression of c-functions  $c_1$  and  $c_2$  vanishes at IR AdS region,  $\Phi$  becomes negative. When  $\Phi$  is negative, the properties of the c-functions  $c_1$  and  $c_2$  become bad, they are not monotonic nor positive, and furthermore they have a singularity in the region given by the solutions in [60, 61]. Thus, for such type of potential other proposal for c-function which is not related with CA should be made.

Hence, we discussed the typical behavior of candidate c-functions. However, it is not clear which role should play dilaton in above expressions as holographic RG coupling constant in dual QFT. It could be induced mass, quantum fields or coupling constants (most probably, gauge coupling), but the explicit rule with what it should be identified is absent. The big number of usual RG parameters in dual QFT suggests also that there should be considered gauged SUGRA with few scalars.

## 4 Surface Counterterms and Finite Action

As well-known, we need to add the surface terms to the bulk action in order to have the well-defined variational principle. Under the variation of the metric  $\hat{G}^{\mu\nu}$  and the scalar field  $\phi$ , the variation of the action (2.1) is given by

$$\begin{aligned} \delta S &= \delta S_{M_{d+1}} + \delta S_{M_d} \\ \delta S_{M_{d+1}} &= \frac{1}{16\pi G} \int_{M_{d+1}} d^{d+1}x \sqrt{-\hat{G}} \left[ \delta \hat{G}^{\zeta\xi} \left\{ -\frac{1}{2} G_{\zeta\xi} \left\{ \hat{R} \right. \right. \right. \end{aligned} \quad (4.1)$$

$$\begin{aligned}
& + (X(\phi) - Y'(\phi)) (\hat{\nabla}\phi)^2 + \Phi(\phi) + 4\lambda^2 \} + \hat{R}_{\zeta\xi} + (X(\phi) - Y'(\phi)) \partial_\zeta \phi \partial_\xi \phi \} \\
& + \delta\phi \left\{ (X'(\phi) - Y''(\phi)) (\hat{\nabla}\phi)^2 + \Phi'(\phi) \right. \\
& \left. - \frac{1}{\sqrt{-\hat{G}}} \partial_\mu \left( \sqrt{-\hat{G}} \hat{G}^{\mu\nu} (X(\phi) - Y'(\phi)) \partial_\nu \phi \right) \right\} \Bigg] . \\
\delta S_{M_d} &= \frac{1}{16\pi G} \int_{M_d} d^d x \sqrt{-\hat{g}} n_\mu \left[ \partial^\mu (\hat{G}_{\xi\nu} \delta \hat{G}^{\xi\nu}) - D_\nu (\delta \hat{G}^{\mu\nu}) + Y(\phi) \partial^\mu (\delta\phi) \right] .
\end{aligned}$$

Here  $\hat{g}_{\mu\nu}$  is the metric induced from  $\hat{G}_{\mu\nu}$  and  $n_\mu$  is the unit vector normal to  $M_d$ . The surface term  $\delta S_{M_d}$  of the variation contains  $n^\mu \partial_\mu (\delta \hat{G}^{\xi\nu})$  and  $n^\mu \partial_\mu (\delta\phi)$ , which makes the variational principle ill-defined. In order that the variational principle is well-defined on the boundary, the variation of the action should be written as

$$\delta S_{M_d} = \lim_{\rho \rightarrow 0} \int_{M_d} d^d x \sqrt{-\hat{g}} \left[ \delta \hat{G}^{\xi\nu} \{ \dots \} + \delta\phi \{ \dots \} \right] \quad (4.2)$$

after using the partial integration. If we put  $\{ \dots \} = 0$  for  $\{ \dots \}$  in (4.2), one could obtain the boundary condition corresponding to Neumann boundary condition. We can, of course, select Dirichlet boundary condition by choosing  $\delta \hat{G}^{\xi\nu} = \delta\phi = 0$ , which is natural for AdS/CFT correspondence. The Neumann type condition becomes, however, necessary later when we consider the black hole mass etc. by using surface terms. If the variation of the action on the boundary contains  $n^\mu \partial_\mu (\delta \hat{G}^{\xi\nu})$  or  $n^\mu \partial_\mu (\delta\phi)$ , however, we cannot partially integrate it on the boundary in order to rewrite the variation in the form of (4.2) since  $n_\mu$  expresses the direction perpendicular to the boundary. Therefore the “minimum” of the action is ambiguous. Such a problem was well studied in [34] for the Einstein gravity and the boundary term was added to the action. It cancels the term containing  $n^\mu \partial_\mu (\delta \hat{G}^{\xi\nu})$ . We need to cancel also the term containing  $n^\mu \partial_\mu (\delta\phi)$ . Then one finds the boundary term [24]

$$S_b^{(1)} = -\frac{1}{8\pi G} \int_{M_d} d^d x \sqrt{-\hat{g}} [D_\mu n^\mu + Y(\phi) n_\mu \partial^\mu \phi] . \quad (4.3)$$

We also need to add surface counterterm  $S_b^{(2)}$  which cancels the divergence coming from the infinite volume of the bulk space, say AdS. In order to investigate the divergence, we choose the metric in the form (2.5). In the parametrization (2.5),  $n^\mu$  and the curvature  $R$  are given by

$$\begin{aligned}
n^\mu &= \left( \frac{2\rho}{l}, 0, \dots, 0 \right) \\
R &= \tilde{R} + \frac{3\rho^2}{l^2} \hat{g}^{ij} \hat{g}^{kl} \hat{g}'_{ik} \hat{g}'_{jl} - \frac{4\rho^2}{l^2} \hat{g}^{ij} \hat{g}''_{ij} - \frac{\rho^2}{l^2} \hat{g}^{ij} \hat{g}^{kl} \hat{g}'_{ij} \hat{g}'_{kl} .
\end{aligned} \quad (4.4)$$

Here  $\tilde{R}$  is the scalar curvature defined by  $g_{ij}$  in (2.5). Expanding  $g_{ij}$  and  $\phi$  with respect to  $\rho$  as in (2.7), we find the following expression for  $S + S_b^{(1)}$ :

$$S + S_b^{(1)} = \frac{1}{16\pi G} \lim_{\rho \rightarrow 0} \int d^d x l \rho^{-\frac{d}{2}} \sqrt{-g_{(0)}} \left[ \frac{2-2d}{l^2} - \frac{1}{d} \Phi(\phi_{(0)}) \right]$$

$$\begin{aligned}
& +\rho \left\{ -\frac{1}{d-2}R_{(0)} - \frac{1}{l^2}g_{(0)}^{ij}g_{(1)ij} \right. \\
& -\frac{1}{d-2}\left( X(\phi_{(0)})\left(\nabla\phi_{(0)}\right)^2 + Y(\phi_{(0)})\Delta\phi_{(0)} \right. \\
& \left. \left. +\Phi'(\phi_{(0)})\phi_{(1)}\right)\right\} + \mathcal{O}\left(\rho^2\right) \Big] . \tag{4.5}
\end{aligned}$$

Then for  $d = 2$

$$S_b^{(2)} = \frac{1}{16\pi G} \int d^d x \sqrt{-\hat{g}} \left[ \frac{2}{l} + \frac{l}{2}\Phi(\phi) \right] \tag{4.6}$$

and for  $d = 3, 4$ ,

$$\begin{aligned}
S_b^{(2)} = \frac{1}{16\pi G} \int d^d x \left[ \sqrt{-\hat{g}} \left\{ \frac{2d-2}{l} + \frac{l}{d-2}R - \frac{2l}{d(d-2)}\Phi(\phi) \right. \right. \\
\left. \left. + \frac{l}{d-2}\left( X(\phi)\left(\hat{\nabla}\phi\right)^2 + Y(\phi)\hat{\Delta}\phi\right) \right\} - \frac{l^2}{d(d-2)}n^\mu\partial_\mu\left(\sqrt{-\hat{g}}\Phi(\phi)\right) \right] . \tag{4.7}
\end{aligned}$$

Note that the last term in above expression does not look typical from the AdS/CFT point of view. The reason is that it does not depend from only the boundary values of the fields. Its presence may indicate to breaking of AdS/CFT conjecture in the situations when SUGRA scalars significantly deviate from constants or are not asymptotic constants.

Here  $\hat{\Delta}$  and  $\hat{\nabla}$  are defined by using  $d$ -dimensional metric and we used

$$\begin{aligned}
\sqrt{-\hat{g}}\Phi(\phi) &= \rho^{-\frac{d}{2}}\sqrt{-g_{(0)}}\left\{ \Phi(\phi_{(0)}) \right. \\
&+ \rho \left( \frac{1}{2}g_{(0)}^{ij}g_{(1)ij}\Phi(\phi_{(0)}) + \Phi'(\phi_{(0)})\phi_{(1)} \right) + \mathcal{O}\left(\rho^2\right) \Big\} \\
n^\mu\partial_\mu\left(\sqrt{-\hat{g}}\Phi(\phi)\right) &= \frac{2}{l}\rho^{-\frac{d}{2}}\sqrt{-g_{(0)}}\left\{ -\frac{d}{2}\Phi(\phi_{(0)}) \right. \\
&+ \rho \left( 1 - \frac{d}{2} \right) \left( \frac{1}{2}g_{(0)}^{ij}g_{(1)ij}\Phi(\phi_{(0)}) + \Phi'(\phi_{(0)})\phi_{(1)} \right) + \mathcal{O}\left(\rho^2\right) \Big\} . \tag{4.8}
\end{aligned}$$

Note that  $S_b^{(2)}$  in (4.6) or (4.7) is only given in terms of the boundary quantities except the last term in (4.7). The last term is necessary to cancel the divergence of the bulk action and it is, of course, the total derivative in the bulk theory:

$$\int d^d x n^\mu \partial_\mu \left( \sqrt{-\hat{g}} \Phi(\phi) \right) = \int d^{d+1} x \sqrt{-\hat{G}} \square \Phi(\phi) . \tag{4.9}$$

Thus we got the boundary counterterm action for gauged SUGRA. Using these local surface counterterms as part of complete action one can show explicitly that bosonic sector of gauged SUGRA in dimensions under discussion gives finite action in asymptotically AdS space. The corresponding example will be given in next section.

Recently the surface counterterms for the action with the dilaton (scalar) potential are discussed in [43]. Their counterterms seem to correspond to the terms cancelling the leading divergence when  $\rho \rightarrow 0$  in (4.5). However, they seem to have only considered the

case where the dilaton becomes asymptotically constant  $\phi \rightarrow \phi_{(0)}$ . If we choose  $\phi_{(0)} = 0$ , the total dilaton potential including the cosmological term  $\mathcal{V}_{\text{dilaton}}(\phi) \equiv 4\lambda^2 + \Phi(\phi)$  approaches to  $\mathcal{V}_{\text{dilaton}}(\phi) \rightarrow 4\lambda^2 = d(d-1)/l^2$ . Then if we only consider the leading  $\rho$  behavior and the asymptotically constant dilaton, the counterterm action in (4.6) and/or (4.7) has the following form

$$S_b^{(2)} = \frac{1}{16\pi G} \int d^d x \sqrt{-\hat{g}} \left( \frac{2d-2}{l} \right), \quad (4.10)$$

which coincides with the result in [43] when the spacetime is asymptotically AdS.

Let us turn now to the discussion of deep connection between surface counterterms and holographic CA. It is enough to mention only  $d = 4$ . In order to control the logarithmically divergent terms in the bulk action  $S$ , we choose  $d - 4 = \epsilon < 0$ . Then

$$S + S_b = \frac{1}{\epsilon} S_{\text{ln}} + \text{finite terms}. \quad (4.11)$$

Here  $S_{\text{ln}}$  is given in (2.14). We also find

$$g_{(0)}^{ij} \frac{\delta}{\delta g_{(0)}^{ij}} S_{\text{ln}} = -\frac{\epsilon}{2} \mathcal{L}_{\text{ln}} + \mathcal{O}(\epsilon^2). \quad (4.12)$$

Here  $\mathcal{L}_{\text{ln}}$  is the Lagrangian density corresponding to  $S_{\text{ln}}$ :  $S_{\text{ln}} = \int d^{d+1} \mathcal{L}_{\text{ln}}$ . Then combining (4.11) and (4.12), we obtain the trace anomaly:

$$T = \lim_{\epsilon \rightarrow 0^-} \frac{2\hat{g}_{(0)}^{ij}}{\sqrt{-\hat{g}_{(0)}}} \frac{\delta(S + S_b)}{\delta \hat{g}_{(0)}^{ij}} = -\frac{1}{2} \mathcal{L}_{\text{ln}}, \quad (4.13)$$

which is identical with the result found in (2.8). We should note that the last term in (4.7) does not lead to any ambiguity in the calculation of CA since  $g_{(0)}$  does not depend on  $\rho$ . If we use the equations of motion (2.15), (2.16), (2.17) and (2.18), we finally obtain the expression (2.22) or (A.5). Hence, we found the finite gravitational action (for asymptotically AdS spaces) in 5-dimensions by adding the local surface counterterm. This action correctly reproduces holographic trace anomaly for dual (gauge) theory. In principle, one can also generalize all results for higher dimensions, say, 6-dimensions, etc. With the growth of dimension, the technical problems become more and more complicated as the number of structures in boundary term is increasing.

## 5 Dilatonic AdS Black Hole and its Mass

Let us consider the black hole or “throat” type solution for the equations of the motion (2.2) and (2.3) when  $d = 4$ . The surface term (4.7) may be used for calculation of the finite black hole mass and/or other thermodynamical quantities.

For simplicity, we choose

$$X(\phi) = \alpha \text{ (constant)}, \quad Y(\phi) = 0 \quad (5.1)$$

and we assume the spacetime metric in the following form:

$$ds^2 = -e^{2\rho} dt^2 + e^{2\sigma} dr^2 + r^2 \sum_{i=1}^{d-1} (dx^i)^2 \quad (5.2)$$

and  $\rho$ ,  $\sigma$  and  $\phi$  depend only on  $r$ . The equations (2.2) and (2.3) can be rewritten in the following form:

$$0 = e^{\rho+\sigma} \Phi'(\phi) - 2\alpha (e^{\rho-\sigma} \phi')' \quad (5.3)$$

$$0 = -\frac{1}{3} e^{2\rho} \left( \Phi(\phi) + \frac{12}{l^2} \right) + \left( \rho'' + (\rho')^2 - \rho' \sigma' + \frac{3\rho'}{r} \right) e^{2\rho-2\sigma} \quad (5.4)$$

$$0 = \frac{1}{3} e^{2\sigma} \left( \Phi(\phi) + \frac{12}{l^2} \right) - \rho'' - (\rho')^2 + \rho' \sigma' + \frac{3\sigma'}{r} + \alpha (\phi')^2 \quad (5.5)$$

$$0 = \frac{1}{3} e^{2\sigma} \left( \Phi(\phi) + \frac{12}{l^2} \right) r^2 + k + \{r(\sigma' - \rho') - 2\} e^{-2\sigma} . \quad (5.6)$$

Here  $' \equiv \frac{d}{dr}$ . If one defines new variables  $U$  and  $V$  by

$$U = e^{\rho+\sigma} , \quad V = r^2 e^{\rho-\sigma} , \quad (5.7)$$

we obtain the following equations from (5.3-5.6):

$$0 = r^3 U \Phi'(\phi) - 2\alpha (rV \phi')' \quad (5.8)$$

$$0 = \frac{1}{3} e^{2\sigma} \left( \Phi(\phi) + \frac{12}{l^2} \right) r^3 U + kr - V' \quad (5.9)$$

$$0 = \frac{3U'}{rU} + \alpha (\phi)' . \quad (5.10)$$

We should note that only three equations in (5.3-5.6) are independent. There is practical problem in the construction of AdS BH with non-trivial dilaton, especially for arbitrary dilatonic potential. That is why we use below the approximate technique which was developed in ref.[7] for constant dilatonic potential.

When  $\Phi(0) = \Phi'(0) = \phi = 0$ , a solution corresponding to the throat limit of D3-brane is given by

$$U = 1 , \quad V = V_0 \equiv \frac{r^4}{l^2} - \mu . \quad (5.11)$$

In the following, we use large  $r$  expansion and consider the perturbation around (5.11). It is assumed

$$\Phi(\phi) = \tilde{\mu} \phi^2 + \mathcal{O}(\phi^3) . \quad (5.12)$$

Then one can neglect the higher order terms in (5.12). We obtain from (5.8)

$$0 \sim \tilde{\mu} r^3 \phi + \alpha \left( \frac{r^5}{l^2} \phi' \right)' . \quad (5.13)$$

The solution of eq.(5.13) is given by

$$\phi = cr^{-\beta} , \quad (c \text{ is a constant}) , \quad \beta = 2 \pm \sqrt{4 - \frac{\tilde{\mu} l^2}{\alpha}} . \quad (5.14)$$

Consider  $r$  is large or  $c$  is small, and write  $U$  and  $V$  in the following form:

$$U = 1 + c^2 u, \quad V = V_0 + c^2 v. \quad (5.15)$$

Then from (5.9) and (5.10), one gets

$$u = u_0 + \frac{\alpha\beta}{6} r^{-2\beta}, \quad v = v_0 - \frac{\tilde{\mu}(\beta-6)}{6(\beta-4)(\beta-2)} r^{-2\beta+4}. \quad (5.16)$$

Here  $u_0$  and  $v_0$  are constants of the integration. Here we choose

$$v_0 = u_0 = 0. \quad (5.17)$$

The horizon which is defined by

$$V = 0 \quad (5.18)$$

lies at

$$r = r_h \equiv l^{\frac{1}{2}} \mu^{\frac{1}{4}} + c^2 \frac{\tilde{\mu}(\beta-6) l^{\frac{5}{2}-\beta} \mu^{\frac{1}{4}-\frac{\beta}{2}}}{24(\beta-4)(\beta-2)}. \quad (5.19)$$

And the Hawking temperature is

$$\begin{aligned} T &= \frac{1}{4\pi} \left[ \frac{1}{r^2} \frac{dV}{dr} \right]_{r=r_h} \\ &= \frac{1}{4\pi} \left\{ 4l^{-\frac{3}{2}} \mu^{\frac{1}{4}} + c^2 \frac{\tilde{\mu}(\beta-6)(2\beta-3)}{6(\beta-4)(\beta-2)} l^{\frac{1}{2}-\beta} \mu^{\frac{1}{4}-\frac{\beta}{2}} \right\}. \end{aligned} \quad (5.20)$$

We now evaluate the free energy of the black hole within the standard prescription [62, 63]. The free energy  $F$  can be obtained by substituting the classical solution into the action  $S$ :

$$F = TS. \quad (5.21)$$

Here  $T$  is the Hawking temperature. Using the equations of motion in (2.2) ( $X = \alpha$ ,  $Y = 0$ ,  $4\lambda^2 = \frac{12}{l^2}$ ), we obtain

$$0 = \frac{5}{3} \left( \Phi(\phi) + \frac{12}{l^2} \right) + \hat{R} + \alpha (\nabla\phi)^2. \quad (5.22)$$

Substituting (5.22) into the action (2.1) after Wick-rotating it to the Euclid signature

$$\begin{aligned} S &= \frac{1}{16\pi G} \cdot \frac{2}{3} \int_{M_5} d^5\sqrt{G} \left( \Phi(\phi) + \frac{12}{l^2} \right) \\ &= \frac{1}{16\pi G} \cdot \frac{2}{3} \frac{V_{(3)}}{T} \int_{r_h}^{\infty} dr r^3 U \left( \Phi(\phi) + \frac{12}{l^2} \right). \end{aligned} \quad (5.23)$$

Here  $V_{(3)}$  is the volume of the 3-dimensional space ( $\int d^5x \cdots = \beta V_{(3)} \int dr r^3 \cdots$ ) and  $\beta$  is the period of time, which can be regarded as the inverse of the temperature  $T$ , ( $\frac{1}{T}$ ). The expression (5.23) contains the divergence. We regularize the divergence by replacing

$$\int_{r_h}^{\infty} dr \rightarrow \int_{r_h}^{r_{\max}} dr \quad (5.24)$$

and subtract the contribution from a zero temperature solution, where we choose  $\mu = c = 0$ , and the solution corresponds to the vacuum or pure AdS:

$$S_0 = \frac{1}{16\pi G} \cdot \frac{2}{3} \cdot \frac{12}{l^2} \frac{V_{(3)}}{T} \sqrt{\frac{G_{tt}(r=r_{\max}, \mu=c=0)}{G_{tt}(r=r_{\max})}} \int_{r_h}^{\infty} dr r^3. \quad (5.25)$$

The factor  $\sqrt{\frac{G_{tt}(r=r_{\max}, \mu=c=0)}{G_{tt}(r=r_{\max})}}$  is chosen so that the proper length of the circles which correspond to the period  $\frac{1}{T}$  in the Euclid time at  $r_{\max}$  coincides with each other in the two solutions. Then we find the following expression for the free energy,

$$\begin{aligned} F &= \lim_{r_{\max} \rightarrow \infty} T(S - S_0) \\ &= \frac{V_{(3)}}{2\pi G l^2 T^2} \left[ -\frac{l^2 \mu}{8} + c^2 \mu^{1-\frac{\beta}{2}} \tilde{\mu} \left\{ \frac{(\beta-1)}{12\beta(\beta-4)(\beta-2)} \right\} + \dots \right]. \end{aligned} \quad (5.26)$$

Here we assume  $\beta > 2$  or the expression  $S - S_0$  still contains the divergences and we cannot get finite results. However, the inequality  $\beta > 2$  is not always satisfied in the gauged SUGRA models. In that case the expression in (5.26) would not be valid. One can express the free energy  $F$  in (5.26) in terms of the temperature  $T$  instead of  $\mu$ :

$$F = \frac{V_{(3)}}{16\pi G} \left[ -\pi T^4 l^6 + c^2 l^{8-4\beta} T^{4-2\beta} \tilde{\mu} \left( \frac{2\beta^3 - 15\beta^2 + 22\beta - 4}{6\beta(\beta-4)(\beta-2)} \right) + \dots \right]. \quad (5.27)$$

Then the entropy  $\mathcal{S}$  and the energy (mass)  $E$  is given by

$$\begin{aligned} \mathcal{S} &= -\frac{dF}{dT} = \frac{V_{(3)}}{16\pi G} \left[ 4\pi T^3 l^6 \right. \\ &\quad \left. + c^2 l^{8-4\beta} T^{3-2\beta} \tilde{\mu} \left( \frac{2\beta^3 - 15\beta^2 + 22\beta - 4}{3\beta(\beta-4)} \right) + \dots \right] \\ E &= F + TS = \frac{V_{(3)}}{16\pi G} \left[ 3\pi T^4 l^6 \right. \\ &\quad \left. + c^2 l^{8-4\beta} (\pi T^4)^{1-\frac{\beta}{2}} \tilde{\mu} \left( \frac{(2\beta-3)(2\beta^3 - 15\beta^2 + 22\beta - 4)}{6\beta(\beta-4)(\beta-2)} \right) + \dots \right]. \end{aligned} \quad (5.28)$$

We now evaluate the mass using the surface term of the action in (4.7), i.e. within local surface counterterm method. The surface energy momentum tensor  $T_{ij}$  is now defined by ( $d=4$ )<sup>5</sup>

$$\delta S_b^{(2)} = \sqrt{-\hat{g}} \delta \hat{g}^{ij} T_{ij}$$

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<sup>5</sup>  $S$  does not contribute due to the equation of motion in the bulk. The variation of  $S + S_b^{(1)}$  gives a contribution proportional to the extrinsic curvature  $\theta_{ij}$  at the boundary:

$$\delta (S + S_b^{(2)}) = \frac{\sqrt{-\hat{g}}}{16\pi G} (\theta_{ij} - \theta \hat{g}_{ij}) \delta \hat{g}^{ij}$$

The contribution is finite even in the limit of  $r \rightarrow \infty$ . Then the finite part does not depend on the parameters characterizing the black hole. Therefore after subtracting the contribution from the reference metric, which could be that of AdS, the contribution from the variation of  $S + S_b^{(1)}$  vanishes.



$$\begin{aligned}
&= \frac{1}{16\pi G} \left[ \sqrt{-\hat{g}} \delta \hat{g}^{ij} \left\{ -\frac{1}{2} \hat{g}_{ij} \left( \frac{6}{l} + \frac{l}{2} \hat{R} + \frac{l}{4} \Phi(\phi) \right) \right\} \right. \\
&\quad \left. + \frac{l^2}{4} n^\mu \partial_\mu \left\{ \sqrt{-\hat{g}} \delta \hat{g}^{ij} \hat{g}_{ij} \Phi(\phi) \right\} \right] . \tag{5.29}
\end{aligned}$$

Note that the energy-momentum tensor is still not well-defined due to the term containing  $n^\mu \partial_\mu$ . If we assume  $\delta \hat{g}^{ij} \sim \mathcal{O}(\rho^{a_1})$  for large  $\rho$  when we choose the coordinate system (2.5), then

$$n^\mu \partial_\mu (\delta \hat{g}^{ij}) \sim \frac{2}{l} \delta \hat{g}^{ij} (a_1 + \partial_\rho)(\cdot) . \tag{5.30}$$

Or if  $\delta \hat{g}^{ij} \sim \mathcal{O}(r^{a_2})$  for large  $r$  when we choose the coordinate system (5.2), then

$$n^\mu \partial_\mu (\delta \hat{g}^{ij}) \sim \delta \hat{g}^{ij} e^\sigma \left( \frac{a_2}{r} + \partial_r \right) (\cdot) . \tag{5.31}$$

As we consider the black hole-like object in this section, one chooses the coordinate system (5.2) and assumes eq.(5.31). Then mass  $E$  of the black hole like object is given by

$$E = \int d^{d-1}x \sqrt{\tilde{\sigma}} N \delta T_{tt} (u^t)^2 . \tag{5.32}$$

Here we assume the metric of the reference spacetime (for example, AdS) has the form of  $ds^2 = f(r)dr^2 - N^2(r)dt^2 + \sum_{i,j=1}^{d-1} \tilde{\sigma}_{ij} dx^i dx^j$  and  $\delta T_{tt}$  is the difference of the  $(t, t)$  component of the energy-momentum tensor in the spacetime with black hole like object from that in the reference spacetime, which we choose to be AdS, and  $u^t$  is the  $t$  component of the unit time-like vector normal to the hypersurface given by  $t = \text{constant}$ . By using the solution in (5.15) and (5.16), the  $(t, t)$  component of the energy-momentum tensor in (5.29) has the following form:

$$\begin{aligned}
T_{tt} = & \frac{3}{16\pi G} \frac{r^2}{l^3} \left[ 1 - \frac{l^3 \mu}{r^4} + l^2 \tilde{\mu} c^2 \left( \frac{1}{12} - \frac{1}{6\beta(\beta-6)} \right. \right. \\
& \left. \left. - \frac{\beta-6}{6(\beta-4)(\beta-2)} - \frac{(3-\beta)(1+a_2)}{12} \right) r^{-2\beta} + \dots \right] . \tag{5.33}
\end{aligned}$$

If we assume the mass is finite,  $\beta$  should satisfy the inequality  $\beta > 2$ , as in the case of the free energy in (5.26) since  $\sqrt{\tilde{\sigma}} N (u^t)^2 = l r^2$  for the reference AdS space. Then the  $\beta$ -dependent term in (5.33) does not contribute to the mass and one gets

$$E = \frac{3\mu V_{(3)}}{16\pi G} . \tag{5.34}$$

Using (5.20)

$$E = \frac{3l^6 V_{(3)} \pi T^4}{16\pi G} \left\{ 1 - c^2 \tilde{\mu} l^{2-4\beta} (\pi T^4)^{-\frac{\beta}{2}} \frac{(\beta-6)(2\beta-3)}{(\beta-4)(\beta-2)} \right\} , \tag{5.35}$$

which does not agree with the result in (5.28). This might express the ambiguity in the choice of the regularization to make the finite action. A possible origin of it might be

following. We assumed  $\phi$  can be expanded in the (integer) power series of  $\rho$  in (2.7) when deriving the surface terms in (4.7). However, this assumption seems to conflict with the classical solution in (5.14), where the fractional power seems to appear since  $r^2 \sim \frac{1}{\rho}$ . In any case, in QFT there is no problem in regularization dependence of the results. In many cases (see example in ref.[41]) the explicit choice of free parameters of regularization leads to coincidence of the answers which look different in different regularizations. As usually happens in QFT the renormalization is more universal as the same answers for beta-functions may be obtained while using different regularizations. That suggests that holographic RG should be developed and the predictions of above calculations should be tested in it.

As in the case of the c-function, we might be drop the terms containing  $\Phi'$  in the expression of  $S_b^{(2)}$  in (4.7). Then we obtain

$$S_b^{(2)} = \frac{1}{16\pi G} \int d^d x \left[ \sqrt{-\hat{g}} \left\{ \frac{2d-2}{l} + \frac{l}{d-2} R + \frac{2l}{d(d-2)} \Phi(\phi) + \frac{l}{d-2} \left( X(\phi) (\hat{\nabla}\phi)^2 + Y(\phi) \hat{\Delta}\phi \right) \right\} - \frac{l^2 \Phi(\phi)}{d(d-2)} n^\mu \partial_\mu \left( \sqrt{-\hat{g}} \right) \right]. \quad (5.36)$$

If we use the expression (5.36), however, the result of the mass  $E$  in (5.35) does not change.

## 6 Gauged Supergravity with Maximally SUSY

In this section, we investigate 2 and 4-dimensional CA (where the bulk scalars potential is included) from 3 and 5-dimensional gauged SUGRA with maximally SUSY, respectively. The only condition is that parametrization of scalar coset is done so that kinetic term for scalars has the standard field theory form. The bulk potential is arbitrary subject to consistent parametrization. So then, we consider the case that includes  $N$  scalars and the coefficients  $X = -\frac{1}{2}$ ,  $Y = 0$ . The bosonic sector of action in this case is

$$S = \frac{1}{16\pi G} \int_{M_{d+1}} d^{d+1}x \sqrt{-\hat{G}} \left\{ \hat{R} - \sum_{\alpha=1}^N \frac{1}{2} (\hat{\nabla}\phi_\alpha)^2 + \Phi(\phi_1, \dots, \phi_N) + 4\lambda^2 \right\}, \quad (6.1)$$

instead of (2.1). The equations of motion are given by variation of (6.1) with respect to  $\phi_\alpha$  and  $G^{\mu\nu}$  as

$$0 = -\sqrt{-\hat{G}} \frac{\partial \Phi(\phi_1, \dots, \phi_N)}{\partial \phi_\beta} - \partial_\mu \left( \sqrt{-\hat{G}} \hat{G}^{\mu\nu} \partial_\nu \phi_\beta \right) \quad (6.2)$$

$$0 = \frac{1}{d-1} \hat{G}_{\mu\nu} \left( \Phi(\phi) + \frac{d(d-1)}{l^2} \right) + \hat{R}_{\mu\nu} - \sum_{\alpha=1}^N \frac{1}{2} \partial_\mu \phi_\alpha \partial_\nu \phi_\alpha. \quad (6.3)$$

One expands  $\phi_\alpha$  with respect to  $\rho$  in a same way as in (2.7).

$$g_{ij} = g_{(0)ij} + \rho g_{(1)ij} + \rho^2 g_{(2)ij} + \dots, \quad \phi_\alpha = \phi_{(0)\alpha} + \rho \phi_{(1)\alpha} + \rho^2 \phi_{(2)\alpha} + \dots. \quad (6.4)$$

$\Phi(\phi_1, \dots, \phi_N)$  is also expanded

$$\begin{aligned} \Phi = & \Phi(\phi_{(0)}) + \rho \sum_{\alpha=1}^N \frac{\partial \Phi(\phi_{(0)})}{\partial \phi_\alpha} \phi_{(1)\alpha} + \rho^2 \left\{ \sum_{\alpha=1}^N \frac{\partial \Phi(\phi_{(0)})}{\partial \phi_\alpha} \phi_{(2)\alpha} \right. \\ & \left. + \frac{1}{2} \sum_{\alpha, \beta=1}^N \frac{\partial^2 \Phi(\phi_{(0)})}{\partial \phi_\alpha \partial \phi_\beta} \phi_{(1)\alpha} \phi_{(1)\beta} \right\} \end{aligned} \quad (6.5)$$

where  $\Phi(\phi_{(0)}) = \Phi(\phi_{(0)1}, \dots, \phi_{(0)N})$ .

We are interested in the SUGRA with maximally SUSY in  $D = d + 1 = 3, 5$  which contain  $N = 128, 42$  scalars respectively (the construction of such 5-dimensional gauged SUGRA has been given in refs.[14, 15]). The maximal SUGRA parameterizes the coset  $E_{11-D}/K$ , where  $E_n$  is the maximally non-compact form of the exceptional group  $E_n$ , and  $K$  is its maximal compact subgroup. The  $SL(N, R)$ , the subgroup of  $E_n$ , can be parameterized via coset  $SL(N, R)/SO(N)$ , and we use the local  $SO(N)$  transformations in order to diagonalize the scalar potential  $\Phi(\phi)$  as in [64, 65, 59]

$$V = \frac{d(d-1)}{N(N-2)} \left( \left( \sum_{i=1}^N X_i \right)^2 - 2 \left( \sum_{i=1}^N X_i^2 \right) \right). \quad (6.6)$$

Let us briefly describe the parametrization leading to the action of form (6.1) given in ref.[64, 65]. Above gauged SUGRA case means that in  $D = 4, 5$  we should take  $N = 8, 6$  respectively.  $N$  scalars  $X_i$  which are constrained by

$$\prod_{i=1}^N X_i = 1 \quad (6.7)$$

can be parameterized in terms of  $(N-1)$  independent dilatonic scalars  $\phi_\alpha$  as follows

$$X_i = e^{-\frac{1}{2} b_i^\alpha \phi_\alpha} \quad (6.8)$$

Here  $b_i^\alpha$  are the weight vectors of the fundamental representation of  $SL(N, R)$ , which satisfy

$$b_i^\alpha b_j^\alpha = 8\delta_{ij} - \frac{8}{N}, \quad \sum_i b_i^\alpha = 0. \quad (6.9)$$

Then the potential has minimum at  $X_i = 1$  ( $N > 5$ ) at the point  $\phi_\alpha = 0$  and  $V = d(d-1)$ . The second derivatives of the potential at this minimum are given by

$$\frac{\partial^2 \Phi(\phi_{(0)})}{\partial \phi_\alpha \partial \phi_\beta} = \frac{d(d-1)}{N(N-2)} b_i^\alpha b_i^\beta \quad (6.10)$$

Here

$$b_i^\alpha b_i^\beta = 4(N-4)\delta^{\alpha\beta}, \quad (6.11)$$

For gauged SUGRAs with maximally SUSY described above (i.e. in  $D = 4, 5$  we take  $N = 8, 6$  respectively), we can get the second derivatives of the potential as

$$\frac{\partial^2 \Phi(\phi_{(0)})}{\partial \phi_\alpha \partial \phi_\beta} = 2(d-2)\delta^{\alpha\beta}. \quad (6.12)$$

The first derivatives of the potential are restricted by the leading order term in the equations of motion (6.2)

$$\frac{\partial \Phi(\phi_{(0)})}{\partial \phi_\alpha} = 0. \quad (6.13)$$

We will use (6.12), (6.13) in the calculations later, but we also consider the case  $\Phi(\phi_{(0)}) = 0$  which corresponds to the constant cosmological term. Then, we introduce the parameters  $a$  and  $l$  and rewrite the conditions (6.12), (6.13) as follows:

$$\begin{aligned} \frac{\partial \Phi(\phi_{(0)})}{\partial \phi_\alpha} &= 0 \\ \frac{\partial^2 \Phi(\phi_{(0)})}{\partial \phi_\alpha \partial \phi_\beta} &= \frac{2(d-2)a}{l^2} \delta^{\alpha\beta}. \end{aligned} \quad (6.14)$$

Here  $a = 1$  corresponds to the condition of conformal boundary [59], and  $a = 0$  is the case where cosmological term is constant. In the calculations later, we will use these conditions (6.14). Then,  $\Phi$  is expanded in a simple form

$$\Phi = \Phi(\phi_{(0)}) + \rho^2 \frac{a}{2l^2} \left( \sum_{\alpha=1}^N 2(d-2)\phi_{(1)\alpha}^2 \right) \quad (6.15)$$

Making the explicit calculations, after some work one can get the holographic CA. For example, for holographic  $d = 2$  anomaly one finds

$$\begin{aligned} S_{\text{ln}} &= -\frac{1}{16\pi G} \frac{l}{2} \int d^2x \sqrt{-g_{(0)}} \left\{ R_{(0)} - \sum_{\alpha}^N \frac{1}{2} g_{(0)}^{ij} \partial_i \phi_{(0)\alpha} \partial_j \phi_{(0)\alpha} \right\} \\ &\quad \times \left( \frac{\Phi(\phi_{(0)})}{2} + \frac{2}{l^2} \right) \left( \Phi(\phi_{(0)}) + \frac{2}{l^2} \right)^{-1}. \end{aligned} \quad (6.16)$$

This is CA of dual 2-dimensional QFT theory living on the boundary of (asymptotically) AdS space. It is evaluated via its 3-dimensional gauged SUGRA dual. Note that one can consider any parametrization of scalars in gauged 3-dimensional SUGRA subject to the form of action (6.1). The bulk scalars potential dependence of anomaly is remarkable.

In 4-dimensional case, the calculation of trace anomaly is more involved. The logarithmic term may be found as

$$\begin{aligned} S_{\text{ln}} &= \frac{1}{16\pi G} \int d^4x \sqrt{-g_{(0)}} \left[ -\frac{1}{2l} g_{(0)}^{ij} g_{(0)}^{kl} \left( g_{(1)ij} g_{(1)kl} - g_{(1)ik} g_{(1)jl} \right) \right. \\ &\quad \left. + \frac{l}{2} \left( R_{(0)}^{ij} - \frac{1}{2} g_{(0)}^{ij} R_{(0)} \right) g_{(1)ij} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{l} \sum_{\alpha}^N \phi_{(1)\alpha}^2 - l \sum_{\alpha}^N \frac{1}{2} \phi_{(1)\alpha} \frac{1}{\sqrt{-g_{(0)}}} \partial_i \left( \sqrt{-g_{(0)}} g_{(0)}^{ij} \partial_j \phi_{(0)\alpha} \right) \\
& - \frac{l}{4} \sum_{\alpha}^N \left( g_{(0)}^{ik} g_{(0)}^{jl} g_{(1)kl} - \frac{1}{2} g_{(0)}^{kl} g_{(1)kl} g_{(0)}^{ij} \right) \partial_i \phi_{(0)\alpha} \partial_j \phi_{(0)\alpha} \\
& - \frac{l}{2} \left( \frac{1}{2} g_{(0)}^{ij} g_{(2)ij} - \frac{1}{4} g_{(0)}^{ij} g_{(0)}^{kl} g_{(1)ik} g_{(1)jl} + \frac{1}{8} (g_{(0)}^{ij} g_{(1)ij})^2 \right) \Phi(\phi_{(0)}) \\
& - \frac{a}{l} \sum_{\alpha}^N \phi_{(1)\alpha}^2 \Big].
\end{aligned} \tag{6.17}$$

The conditions (6.14) are used here. The equation of motion (6.3) leads to  $g_{(1)ij}$  in terms of  $g_{(0)ij}$  in the same way as in section 2.

$$\begin{aligned}
g_{(1)ij} &= \left[ -R_{(0)ij} + \sum_{\alpha}^N \frac{1}{2} \partial_i \phi_{(0)\alpha} \partial_j \phi_{(0)\alpha} \right. \\
&+ \frac{g_{(0)ij}}{l^2} \left\{ R_{(0)} - \sum_{\alpha}^N \frac{1}{2} g_{(0)}^{ij} \partial_i \phi_{(0)\alpha} \partial_j \phi_{(0)\alpha} \right\} \times \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{6}{l^2} \right)^{-1} \Big] \\
&\times \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{2}{l^2} \right)^{-1}.
\end{aligned} \tag{6.18}$$

In the equation (6.2), the terms proportional to  $\rho^{-2}$  lead to  $\phi_{(1)}$  as follows:

$$\phi_{(1)\beta} = -\frac{l^2}{4(a-1)} \frac{\partial_i}{\sqrt{-g_{(0)}}} \left( \sqrt{-g_{(0)}} g_{(0)}^{ij} \partial_j \phi_{(0)\beta} \right). \tag{6.19}$$

In the equation (6.3), the terms proportional to  $\rho^1$  with  $\mu, \nu = i, j$  lead to  $g_{(2)ij}$

$$\begin{aligned}
g_{(2)ij} &= \left[ -g_{(0)ij} \frac{2a}{3} \sum_{\alpha}^N \phi_{(1)\alpha}^2 - \frac{2}{l^2} g_{(0)}^{kl} g_{(1)ki} g_{(1)lj} + \frac{1}{l^2} g_{(0)}^{km} g_{(0)}^{nl} g_{(1)mn} g_{(1)kl} g_{(0)ij} \right. \\
&- \frac{2}{l^2} g_{(0)ij} \left( \frac{1}{3} \Phi(\phi_{(0)}) + \frac{8}{l^2} \right)^{-1} \times \left\{ \frac{2}{l^2} g_{(0)}^{mn} g_{(0)}^{kl} g_{(1)km} g_{(1)ln} - \frac{8a}{3} \sum_{\alpha}^N \phi_{(1)\alpha}^2 \right. \\
&\left. \left. + \sum_{\alpha}^N g_{(0)}^{kl} \partial_k \phi_{(1)\alpha} \partial_l \phi_{(0)\alpha} \right\} + \sum_{\alpha}^N \partial_i \phi_{(1)\alpha} \partial_j \phi_{(0)\alpha} \right] \times \left( \frac{1}{3} \Phi(\phi_{(0)}) \right)^{-1}.
\end{aligned} \tag{6.20}$$

Therefore the anomaly term (6.17) is evaluated as

$$\begin{aligned}
S_{ln} &= -\frac{1}{2} \int d^4x \sqrt{-g} T, \\
T &= -\frac{1}{8\pi G} \left[ h_1 R^2 + h_2 R^{ij} R_{ij} + h_3 R^{ij} \sum_{\alpha}^N \partial_i \phi_{(0)\alpha} \partial_j \phi_{(0)\alpha} \right. \\
&+ h_4 R \sum_{\alpha}^N g_{(0)}^{ij} \partial_i \phi_{(0)\alpha} \partial_j \phi_{(0)\alpha} + h_5 \left( \sum_{\alpha}^N g_{(0)}^{ij} \partial_i \phi_{(0)\alpha} \partial_j \phi_{(0)\alpha} \right)^2 \\
&\left. + h_6 \sum_{\alpha}^N \sum_{\beta}^N \left( g_{(0)}^{ij} \partial_i \phi_{(0)\alpha} \partial_j \phi_{(0)\beta} \right)^2 + h_7 \sum_{\alpha}^N \left( \frac{\partial_i}{\sqrt{-g}} \left( \sqrt{-g} g_{(0)}^{ij} \partial_j \phi_{(0)\alpha} \right) \right)^2 \right].
\end{aligned} \tag{6.21}$$

Here  $h_1, h_2, \dots, h_7$  are

$$h_1 = -h_4 = 4h_5 = \frac{3(62208 + 22464\Phi + 2196\Phi^2 + 72\Phi^3 + \Phi^4)l^3}{16(6 + \Phi)^2(18 + \Phi)^2(24 + \Phi)} \quad (6.22)$$

$$h_2 = -h_3 = 4h_6 = -\frac{3(288 + 72\Phi + \Phi^2)l^3}{8(6 + \Phi)^2(24 + \Phi)}$$

$$h_7 = \frac{((a-1)(\Phi+24) - \Phi(a-3))l^3}{16(a-1)^2(\Phi+24)}. \quad (6.23)$$

Hereafter, we denote  $\Phi(\phi_{(0)})$  by  $\Phi$  and do not write the index (0) for the simplicity. We also take  $\Phi \rightarrow l^2\Phi$  as dimensionless, then we can see the dimension of  $h$  easily, i.e. dimension  $h = l^3$ . Thus, we found the holographic CA for QFT dual from 5-dimensional gauged SUGRA with some number of scalars which parameterize the full scalar coset. Note that bulk scalar potential is arbitrary. The only requirement is the form of action (6.1). One can use the explicit parametrization of ref.[64, 65] described above or any other parametrization of 5-dimensional gauged SUGRA leading to the action of form (6.1).

Let us compare now the above CA with already known cases for single scalar. First of all, let us check the condition that the gravitational terms of anomaly (6.21) can be written as a sum of the Gauss-Bonnet invariant  $G$  and the square of the Weyl tensor,  $F$ . They are

$$G = R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl} \quad (6.24)$$

$$F = \frac{1}{3}R^2 - 2R_{ij}R^{ij} + R_{ijkl}R^{ijkl}. \quad (6.25)$$

Then  $R^2$  and  $R_{ij}R^{ij}$  are given by

$$R^2 = 3G - 6F + 3R_{ijkl}R^{ijkl}$$

$$R_{ij}R^{ij} = \frac{1}{2}G - \frac{3}{2}F + R_{ijkl}R^{ijkl}. \quad (6.26)$$

If one can rewrite the anomaly (6.21) as a sum of  $G$  and  $F$ , then  $h_1$  and  $h_2$  satisfy  $3h_1 + h_2 = 0$ . This leads to the following condition for  $\Phi$

$$3h_1 + h_2 = \frac{3\Phi^2(180 + \Phi^2)l^3}{16(6 + \Phi)^2(18 + \Phi)^2(24 + \Phi)} = 0, \quad (6.27)$$

The only solution is  $\Phi = 0$ , i.e. constant bulk potential. In the limit of  $\Phi \rightarrow 0$ , we obtain

$$h_1 \rightarrow \frac{3 \cdot 62208l^3}{16 \cdot 6^2 \cdot 18^2 \cdot 24} = \frac{l^3}{24}$$

$$h_2 \rightarrow -\frac{3 \cdot 288l^3}{8 \cdot 6^2 \cdot 24} = -\frac{l^3}{8}, \quad (6.28)$$

and

$$h_3 \rightarrow +\frac{l^3}{8}, \quad h_4 \rightarrow -\frac{l^3}{24}$$

$$h_5 + h_6 \rightarrow -\frac{l^3}{48} \quad (6.29)$$

If we take the coefficient  $X = -\frac{1}{2}, Y = 0$ , i.e.  $V = -\frac{1}{2}$ ,  $h_3, h_4, h_5 + h_6$  agree with the single scalar case discussed in section 2 exactly. In this limit one gets  $h_7$  as

$$\begin{aligned} h_7 &\rightarrow -\frac{l^3}{16} \quad (a = 0) \\ h_7 &\rightarrow -\infty \cdot l^3 \quad (a = 1), \end{aligned} \quad (6.30)$$

Hence, we find that  $a = 0$  case in  $h_7$  agrees with the result in section 2. Thus, we proved that our trace anomaly coincides with the one for single scalar with constant bulk potential case. It is remarkable that in this case the holographic CA is equal to QFT CA for Yang-Mills theory with maximally SUSY coupled with  $\mathcal{N} = 4$  conformal SUGRA [56].

Now, one considers the case  $a = 1$  which corresponds to the condition [59]. It may look that in this situation the CA contains the divergence. Let us show how to take this limit correctly, so that divergence does not actually appear. For the case of  $a = 1$ , the equation (6.19) becomes

$$\frac{\partial_i}{\sqrt{-g_{(0)}}} \left( \sqrt{-g_{(0)}} g_{(0)}^{ij} \partial_j \phi_{(0)\beta} \right) = 0. \quad (6.31)$$

Therefore we cannot regard  $\phi_{(0)}$  as the degree of freedom on the boundary. Instead of it, we should regard  $\phi_{(1)}$ , which corresponds to  $d\phi/d\rho$  on the boundary, as the independent degree of freedom. The divergence of  $h_7$  at  $a = 1$  should reflect this situation since the divergence prevents us to solve  $\phi_{(1)}$  using  $\phi_{(0)}$ . That is,  $\phi_{(1)}$  becomes independent degree of freedom when  $a = 1$ .

So then, in the case of  $a = 1$ , the anomaly is rewritten in terms of  $\phi_{(0)}, \phi_{(1)}$  as

$$\begin{aligned} T = & -\frac{1}{8\pi G} \left[ h_1 R^2 + h_2 R^{ij} R_{ij} + h_3 R^{ij} \sum_{\alpha}^N \partial_i \phi_{(0)\alpha} \partial_j \phi_{(0)\alpha} \right. \\ & + h_4 R \sum_{\alpha}^N g_{(0)}^{ij} \partial_i \phi_{(0)\alpha} \partial_j \phi_{(0)\alpha} + h_5 \left( \sum_{\alpha}^N g_{(0)}^{ij} \partial_i \phi_{(0)\alpha} \partial_j \phi_{(0)\alpha} \right)^2 \\ & + h_6 \sum_{\alpha}^N \sum_{\beta}^N \left( g_{(0)}^{ij} \partial_i \phi_{(0)\alpha} \partial_j \phi_{(0)\beta} \right)^2 \\ & \left. + \frac{h_7}{l^2} \sum_{\alpha}^N \phi_{(1)\alpha} \frac{\partial_i}{\sqrt{-g}} \left( \sqrt{-g} g_{(0)}^{ij} \partial_j \phi_{(0)\alpha} \right) + \frac{h_8}{l^4} \sum_{\alpha}^N \phi_{(1)\alpha}^2 \right]. \end{aligned} \quad (6.32)$$

Note that from above anomaly one can get the local surface counterterms in the same way as in section 4 and refs.[59]. The coefficients  $h_1, h_2, \dots, h_6$  are the same as for the case  $a \neq 1$  in (6.21).  $h_7$  and  $h_8$  are given by

$$h_7 = \frac{(\Phi - 48)l^3}{4(\Phi + 24)} \quad (6.33)$$

$$h_8 = \frac{2\Phi l^3}{(\Phi + 24)}. \quad (6.34)$$

For the constant dilaton case, eq.(6.32) becomes

$$T = -\frac{1}{8\pi G} \left[ h_1 R^2 + h_2 R^{ij} R_{ij} \right] \quad (6.35)$$

It is interesting to note that coefficients  $h_1, h_2$  which do not depend on number of scalars in above expression may play the role of c-function in UV limit in the same way as in section 3. From the point of view of AdS/CFT correspondence the exponent of scalar should correspond to gauge coupling constant. Hence, this expression represents the (exact) CA with radiative corrections for dual QFT. It is evaluated from SUGRA side. It is non trivial task to get the anomaly for any specific bulk potential.

Hence, we found explicitly non-perturbative CA from gauged SUGRA side in the situation when scalars respect the conformal boundary condition. It corresponds to the one of dual QFT living on the boundary of asymptotically AdS space.

## 7 Scheme Dependence of Conformal Anomaly

In this section, we discuss the scheme dependence of the calculation of CA. We calculated the CA by the method of Henningson-Skenderis [18] in section 2. The point of this method is that the classical AdS-like solutions of 5-dimensional gauged SUGRA after the expansion over radial coordinate can be used to get holographic CA for dual QFT.

Generally, we can express CA for interacting QFT in terms of gravitational invariants multiplied to multi-loop QFT beta-functions (see ref.[46] for recent discussion). One of the features of multi-loop beta-functions for coupling constants is their explicit scheme dependence (or regularization dependence) which normally occurs beyond second loop. Usually, multi-loop quantum calculation is almost impossible to do, the result is known only in couple first orders of loop expansion, hence use of holographic CA is a challenge. Then making calculation of holographic CA which corresponds to dual interacting QFT in different schemes leads also to scheme dependence of such CA.

There are appeared the formulation of holographic RG based on Hamilton-Jacobi approach recently [47] (see also [48, 49]). This formalism permits to find the holographic CA without using the expansion of metric and dilaton over radial coordinate in AdS-like space. The purpose of this section is to calculate holographic CA for multi-dilaton gravity with non-trivial bulk potential in de Boer-Verlinde-Verlinde formalism [47]. Then, the coefficients of curvature as functions of bulk potential are obtained. The comparison of these coefficients (c-functions) with the ones found in section 3 is done. It shows that coefficients coincide only when bulk potential is constant, in other words, holographic CA including non-constant bulk potential is scheme dependent.

We start from the 5-dimensional dilatonic gravity action which is given by

$$S = \frac{1}{2\kappa_5} \int d^5x \sqrt{g} \left[ R + \frac{1}{2} G(\phi) (\nabla \phi)^2 + V(\phi) \right]. \quad (7.1)$$

and choose the 5-dimensional metric in the following form as used in [47, 58]

$$g_{mn} dx^m dx^n = d\rho^2 + \gamma_{\sigma\nu}(\rho, x) dx^\sigma dx^\nu \quad (7.2)$$

Here  $\rho$  is the radial coordinate in AdS-like background. In the following we only consider the case  $G(\phi) = -1$  in (7.1) for simplicity. As in [47, 58], we adopt Hamilton-Jacobi theory. First, we shall cast the 5-dimensional dilatonic gravity action into the canonical



formalism

$$I = \frac{1}{2\kappa_5^2} \int_M d^5x \sqrt{g} \left\{ R + \frac{1}{2} G(\phi) (\nabla\phi)^2 + V(\phi) \right\} \quad (7.3)$$

$$\begin{aligned} &\equiv \frac{1}{2\kappa_5^2} \int d\rho L \\ L &= \int d^4x \sqrt{\gamma} \left[ \pi_{\sigma\nu} \dot{\gamma}^{\sigma\nu} - \Pi \dot{\phi} - \mathcal{H} \right] \end{aligned} \quad (7.4)$$

where  $\dot{\phantom{x}}$  denotes the derivative with respect to  $\rho$ . The canonical momenta and the Hamiltonian density are defined by

$$\begin{aligned} \pi_{\sigma\nu} &\equiv \frac{1}{\sqrt{\gamma}} \frac{\delta L}{\delta \dot{\gamma}^{\sigma\nu}}, \quad \Pi \equiv \frac{1}{\sqrt{\gamma}} \frac{\delta L}{\delta \dot{\phi}} \\ \mathcal{H} &\equiv \frac{1}{3} \pi^2 - \pi_{\sigma\nu} \pi^{\sigma\nu} + \frac{\Pi^2}{2G} - \mathcal{L}, \\ \mathcal{L} &\equiv \mathcal{R} + \frac{1}{2} G \gamma^{\sigma\nu} \partial_\sigma \phi \partial_\nu \phi + V. \end{aligned} \quad (7.5)$$

Here  $\mathcal{R}$  is Ricci scalar of the boundary metric  $\gamma_{\mu\nu}$ . The flow velocity of  $g_{\mu\nu}$  is given by

$$\dot{\gamma}_{\sigma\nu} = 2\pi_{\sigma\nu} - \frac{2}{3} \gamma_{\sigma\nu} \pi^\lambda_\lambda. \quad (7.6)$$

This canonical formulation constraints Hamiltonian as  $\mathcal{H} = 0$ , which leads to the equation

$$\frac{1}{3} \pi^2 - \pi_{\sigma\nu} \pi^{\sigma\nu} + \frac{\Pi^2}{2G} = \mathcal{R} + \frac{1}{2} G \gamma^{\sigma\nu} \partial_\sigma \phi \partial_\nu \phi + V \quad (7.7)$$

Applying de Boer-Verlinde-Verlinde formalism, one can decompose the action  $S[\gamma, \phi]$  in a local and non-local part as follows

$$\begin{aligned} S[\gamma, \phi] &= S_{EH}[\gamma, \phi] + \Gamma[\gamma, \phi] \\ S_{EH}[\gamma, \phi] &= \int d^4x \sqrt{\gamma} \left[ Z(\phi) R + \frac{1}{2} M(\phi) \gamma^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + U(\phi) \right]. \end{aligned} \quad (7.8)$$

Here  $S_{EH}$  is tree level renormalized action and  $\Gamma$  contains the higher-derivative and non-local terms. The canonical momenta are related to the Hamilton-Jacobi functional  $S$  by

$$\pi_{\sigma\nu} = \frac{1}{\sqrt{\gamma}} \frac{\delta S}{\delta \gamma^{\sigma\nu}}, \quad \Pi = \frac{1}{\sqrt{\gamma}} \frac{\delta S}{\delta \phi} \quad (7.9)$$

The expectation value of stress tensor  $\langle T_{\sigma\nu} \rangle$  and that of the gauge invariant operator  $\langle O_\phi \rangle$  which couples to  $\phi$  can be related to  $\Gamma$  by

$$\langle T_{\sigma\nu} \rangle = \frac{2}{\sqrt{\gamma}} \frac{\delta \Gamma}{\delta \gamma^{\sigma\nu}}, \quad \langle O_\phi \rangle = \frac{1}{\sqrt{\gamma}} \frac{\delta \Gamma}{\delta \phi}. \quad (7.10)$$

Then, one can get holographic trace anomaly in the following form

$$\langle T^\mu_\mu \rangle = \beta \langle O_\phi \rangle - c R_{\mu\nu} R^{\mu\nu} + d R^2 \quad (7.11)$$

where  $\beta$  is some beta function and coefficients  $c$  and  $d$  are c-functions. Explicit structure of  $\beta \langle O_\phi \rangle$  is given in section 2:

$$\begin{aligned} \beta \langle O_\phi \rangle = & -2 \left[ h_3 R^{ij} \partial_i \phi \partial_j \phi + h_4 R g^{ij} \partial_i \phi \partial_j \phi + h_5 \frac{R}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right. \\ & + h_6 (g^{ij} \partial_i \phi \partial_j \phi)^2 + h_7 \left( \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right)^2 \\ & \left. + h_8 g^{kl} \partial_k \phi \partial_l \phi \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right] . \end{aligned} \quad (7.12)$$

Here  $h_3 \cdots h_8$  are functions of dilaton  $\phi$ :  $h_i = h_i(\phi)$  ( $i = 3, 4, \dots, 8$ ). To get the explicit forms of c-functions, one substitutes the action (7.8) into (7.9)(7.10) thus one can get the relation between potentials  $U$  and  $V$  by using Hamilton-Jacobi equation (7.7). From the potential term, we get

$$\frac{U^2}{3} + \frac{U'^2}{2G} = V \quad (7.13)$$

and the curvature term  $R$  leads to

$$\frac{U}{3} Z + \frac{U'}{G} Z' = 1. \quad (7.14)$$

where  $'$  denotes the derivative with respect to  $\phi$ . Examining the terms of  $\langle T_{\sigma\nu} \rangle$ , one can get the explicit form of c-functions as

$$c = \frac{6Z^2}{U}, \quad d = \frac{2}{U} \left( Z^2 + \frac{3Z'^2}{2G} \right). \quad (7.15)$$

If we choose constant potential  $V(\phi) = 12$ , by using (7.13) and (7.14), we find  $U$ ,  $Z$  become constant:

$$U = 6, \quad Z = \frac{1}{2} \quad (7.16)$$

Then, c-function  $c$  and  $d$  become

$$c = \frac{1}{4}, \quad d = \frac{1}{12}. \quad (7.17)$$

This exactly reproduces the correspondent coefficients of holographic CA obtained in ref.[18] in the scheme where expansion of 5-dimensional AdS metric in terms of radial AdS coordinate has been adopted. Now we can understand the coincidence of CA calculations between scheme of ref.[18, 24, 56] and de Boer-Verlinde-Verlinde formalism when the scalar potential is constant [66].

At the next step we come to application of above holographic RG formalism in the calculation of CA for 5-dimensional multi-dilaton gravity with non-trivial bulk potential. Such a theory naturally appears as bosonic sector of 5-dimensional gauged supergravity.

We consider gauged SUGRA with maximally SUSY where scalars parameterize a sub-manifold of the full scalar coset [64, 65, 59]. As a result, the bulk potential cannot be chosen arbitrarily. Hence, one limits to the case that includes  $N$  scalars and the coefficient  $G = -1$ . The bosonic sector of the action in this case is

$$S = \frac{1}{16\pi G} \int_{M_D} d^D x \sqrt{-\hat{G}} \left\{ \hat{R} - \sum_{\alpha=1}^N \frac{1}{2} (\hat{\nabla} \phi_\alpha)^2 + V(\phi_1, \dots, \phi_N) \right\}. \quad (7.18)$$

The SUGRA with maximally SUSY in  $D = 5$  contains 42 scalars (the construction of such 5-dimensional gauged SUGRA is given in ref.[14, 15]). The maximal SUGRA parameterizes the coset  $E_{11-D}/K$ , where  $E_n$  is the maximally non-compact form of the exceptional group  $E_n$ , and  $K$  is its maximal compact subgroup. The group  $SL(N, R)$ , a subgroup of  $E_n$ , can be parameterized with the coset  $SL(N, R)/SO(N)$ , and we use the local  $SO(N)$  transformations in order to diagonalize the scalar potential  $V(\phi)$  as in ref.[64, 65]

$$V = \frac{(D-1)(D-2)}{N(N-2)} \left( \left( \sum_{i=1}^N X_i \right)^2 - 2 \left( \sum_{i=1}^N X_i^2 \right) \right). \quad (7.19)$$

Especially for  $D = 5$ , we have

$$V = \frac{1}{2} \left\{ \left( \sum_{i=1}^6 X_i \right)^2 - 2 \left( \sum_{i=1}^6 X_i^2 \right) \right\} \quad (7.20)$$

Let us briefly describe the parameterization leading to the action of form (7.18) given in ref. [64, 65]. In the above gauged SUGRA case, in  $D = 5$  one should set  $N = 6$ . The  $N$  scalars  $X_i$ , which are constrained by

$$\prod_{i=1}^N X_i = 1, \quad (7.21)$$

can be parameterized in terms of  $(N-1)$  independent dilatonic scalars  $\phi_\alpha$  as

$$X_i = e^{-\frac{1}{2} b_i^\alpha \phi_\alpha} \quad (7.22)$$

Here the quantities  $b_i^\alpha$  are the weight vectors of the fundamental representation of  $SL(N, R)$ , which satisfy

$$\begin{aligned} b_i^\alpha b_j^\alpha &= 8\delta_{ij} - \frac{8}{N}, \quad \sum_i b_i^\alpha = 0 \\ b_i^\alpha b_i^\beta &= 4(N-4)\delta^{\alpha\beta}. \end{aligned} \quad (7.23)$$

Then, potential has a minimum at  $X_i = 1$  ( $N > 5$ ) at the point  $\phi_\alpha = 0$ , where  $V = (D-1)(D-2)$ .

To get c-functions  $c$  and  $d$ , one take  $U$  and  $Z$  as

$$\begin{aligned} U &= A \sum_{i=1}^6 e^{-\frac{1}{2} b_i^\alpha \phi_\alpha} = A \sum_{i=1}^6 X_i \\ Z &= B \sum_{i=1}^6 e^{\frac{1}{2} b_i^\alpha \phi_\alpha} = B \sum_{i=1}^6 X_i^{-1} \end{aligned} \quad (7.24)$$

where  $A$  and  $B$  can be determined by the analogues with  $G = -1$  of the conditions (7.13) and (7.14) [66],

$$\frac{U^2}{3} - \sum_{\alpha} \left( \frac{\partial U}{\partial \phi_{\alpha}} \right)^2 = V, \quad \frac{U}{3} Z - \sum_{\alpha} \frac{\partial U}{\partial \phi_{\alpha}} \frac{\partial Z}{\partial \phi_{\alpha}} = 1. \quad (7.25)$$

as follows

$$A = \pm 1, \quad B = \pm \frac{1}{12}. \quad (7.26)$$

Then, using (7.17), c-functions are found

$$\begin{aligned} c &= \frac{6Z^2}{U} = \frac{1}{24} \left( \sum_i^6 X_i^{-1} \right)^2 \left( \sum_j^6 X_j \right)^{-1} \\ d &= \frac{2}{U} \left( Z^2 - \frac{3}{2} \sum_{\alpha} \left( \frac{\partial Z}{\partial \phi_{\alpha}} \right)^2 \right) \\ &= \frac{1}{24} \left( \sum_j^6 X_j \right)^{-1} \left\{ \frac{1}{2} \left( \sum_i^6 X_i^{-1} \right)^2 - \sum_i^6 X_i^{-2} \right\} \end{aligned} \quad (7.27)$$

Thus, the coefficients of gravitational terms in holographic CA (corresponding to dual CFT) from multi-dilaton 5-dimensional gravity are found. The holographic RG formalism is used in such calculation. In [58], the c-functions  $c$  and  $d$  have been found for version of dilaton gravity dual to non-commutative Yang-Mills (NCYM) theory. The gravity theory contains only one dilaton field  $\phi$ . The action can be obtained by putting

$$G(\phi) = -\frac{20}{3\phi^2}, \quad V(\phi) = \frac{1}{\phi^{\frac{8}{3}}} \left( 20 - \frac{8}{\phi^4} \right). \quad (7.28)$$

Then using (7.13), (7.14) and (7.17), one finds

$$c = \frac{\phi^2(\phi^2 + 2)^2}{12(5\phi^2 - 2)}, \quad d = \frac{\phi^2(\phi^4 + 8\phi^2 + 6)}{60(5\phi^2 - 2)}. \quad (7.29)$$

This coincides with the result of ref.[58] and gives useful check of these calculations.

Let us turn now to results of calculation of holographic CA done in section 2 where another scheme [18] was used. In such scheme 5-dimensional metric and scalars are expanded in terms of radial fifth coordinate. Note that as in above evaluation the dilaton and bulk potential are considered to be non-trivial and non-constant.

The functions  $2h_1$  and  $-2h_2$  in notations of section 2 correspond to c-functions  $d$  and  $c$  in (7.27), respectively, and they are given by,

$$h_1 = \frac{3(62208 + 22464\mathcal{V} + 2196\mathcal{V}^2 + 72\mathcal{V}^3 + \mathcal{V}^4)l^3}{16(6 + \mathcal{V})^2(18 + \mathcal{V})^2(24 + \mathcal{V})} \quad (7.30)$$

$$h_2 = -\frac{3(288 + 72\mathcal{V} + \mathcal{V}^2)l^3}{8(6 + \mathcal{V})^2(24 + \mathcal{V})} \quad (7.31)$$

where

$$\mathcal{V} \equiv V(\phi) - V(0) = V(\phi) - 12. \quad (7.32)$$

Since the expressions of  $c$ ,  $d$  seem to be very different from  $2h_1$ ,  $-2h_2$  which are obtained with help of expansion of metric and bulk potential on radial coordinate, we now investigate if they are really different by expanding  $X_i$  on  $\phi$  (up to second order on  $\phi^2$ )

$$\begin{aligned} X_i &= 1 - \frac{1}{2}b_i^\alpha \phi_\alpha + \frac{1}{8}(b_i^\alpha \phi_\alpha)^2 \\ \sum_i^6 X_i &= 6 - \frac{1}{2} \sum_i^6 b_i^\alpha \phi_\alpha + \frac{1}{8} \sum_i^6 (b_i^\alpha \phi_\alpha)^2 \\ (\sum_i^6 X_i)^2 &= 36 - 6 \sum_i^6 b_i^\alpha \phi_\alpha + \frac{3}{2} \sum_i^6 (b_i^\alpha \phi_\alpha)^2 + \frac{1}{4} (\sum_i^6 b_i^\alpha \phi_\alpha)^2 \\ \sum_i^6 X_i^2 &= 6 - \sum_i^6 b_i^\alpha \phi_\alpha + \frac{1}{2} \sum_i^6 (b_i^\alpha \phi_\alpha)^2 \\ X_i^{-1} &= 1 + \frac{1}{2}b_i^\alpha \phi_\alpha - \frac{1}{8}(b_i^\alpha \phi_\alpha)^2 \\ \sum_i^6 X_i^{-1} &= 6 + \frac{1}{2} \sum_i^6 b_i^\alpha \phi_\alpha - \frac{1}{8} \sum_i^6 (b_i^\alpha \phi_\alpha)^2 \\ (\sum_i^6 X_i^{-1})^2 &= 36 + 6 \sum_i^6 b_i^\alpha \phi_\alpha - \frac{3}{2} \sum_i^6 (b_i^\alpha \phi_\alpha)^2 + \frac{1}{4} (\sum_i^6 b_i^\alpha \phi_\alpha)^2 \end{aligned} \quad (7.33)$$

Using (7.22) and (7.23), one finds c-functions  $c$  and  $d$  in (7.27) are given by

$$c = \frac{1}{4} - \frac{1}{64} \sum_i^6 (b_i^\alpha \phi_\alpha)^2 \quad (7.34)$$

$$d = \frac{1}{12} - \frac{1}{144} \sum_i^6 (b_i^\alpha \phi_\alpha)^2 \quad (7.35)$$

$h_1$  and  $h_2$  in (7.30) are given by

$$h_1 = \frac{1}{2} \left( \frac{1}{12} - \frac{1}{384} \sum_i^6 (b_i^\alpha \phi_\alpha)^2 \right) \quad (7.36)$$

$$-h_2 = \frac{1}{2} \left( \frac{1}{4} - \frac{1}{128} \sum_i^6 (b_i^\alpha \phi_\alpha)^2 \right). \quad (7.37)$$

Then  $2h_1$  and  $-2h_2$  do not coincide with  $d$  and  $c$ , except the leading constant part. One finds the formalism by de Boer-Verlinde-Verlinde [47] does not reproduce the result based on the scheme of ref.[18]. Technically, this disagreement might occur since we expand dilatonic potential in the power series on  $\rho$  and this is the reason of ambiguity and scheme dependence of holographic CA.

This result means that holographic CA (with non-trivial bulk potential and non-constant dilaton) is scheme dependent. AdS/CFT correspondence says that such holographic CA should correspond to (multi-loop) QFT CA (dilaton plays the role of coupling constants). However, QFT multi-loop CA depends on regularization (as beta-functions are also scheme dependent). Hence, scheme dependence of holographic CA is consistent with QFT expectations. That also means that two different formalisms we discussed in this thesis actually should correspond to different regularizations of dual QFT. We also discuss 2-dimensional CA case (i.e. AdS<sub>3</sub>/CFT<sub>2</sub>) in Appendix D.

## 8 AdS<sub>9</sub>/CFT<sub>8</sub> Correspondence

In this section we consider the extension of the Hamilton-Jacobi formalism to higher dimensions which provides the interesting formulation of holographic RG. This section is based on the most recent work [50].

It is known quite a lot about AdS/CFT correspondence in dimensions below 8, say, about AdS<sub>7</sub>/CFT<sub>6</sub>, AdS<sub>5</sub>/CFT<sub>4</sub> or AdS<sub>3</sub>/CFT<sub>2</sub> set-up (see review [13] and refs. therein). It would be of great interest to extend the corresponding results to higher dimensions: 9 and 11-dimensions (where M-theory is presumably residing). As one step in this direction one can calculate the holographic CA in higher dimensions. In the present section, starting from the systematic prescription for solving the Hamilton-Jacobi equation (i.e. flow equation) given in [66], we will perform such a calculation in 8-dimensions.<sup>6</sup> The calculation of 8-dimensional CA is very complicate (the 6-dimensional case is also complicate) thus, we note those calculations in Appendix E. First, we briefly review the formulation discussed in [47, 66]. One starts from  $d + 1$ -dimensional AdS-like metric in the following form

$$ds^2 = G_{MN}dX^M dX^N = dr^2 + G_{\mu\nu}(x, r)dx^\mu dx^\nu. \quad (8.1)$$

where  $X^M = (x^\mu, r)$  with  $\mu, \nu = 1, 2, \dots, d$ . The action on a  $(d + 1)$ -dimensional manifold  $M_{d+1}$  with the boundary  $\Sigma_d = \partial M_{d+1}$  is given by

$$\begin{aligned} S_{d+1} &= \int_{M_{d+1}} d^{d+1}x \sqrt{G} (V - R) - 2 \int_{\Sigma_d} d^d x \sqrt{G} K \\ &= \int_{\Sigma_d} d^d x \int dr \sqrt{G} (V - R + K_{\mu\nu} K^{\mu\nu} - K^2) \\ &\equiv \int d^d x dr \sqrt{G} \mathcal{L}_{d+1}. \end{aligned} \quad (8.2)$$

where  $R$  and  $K_{\mu\nu}$  are the scalar curvature and the extrinsic curvature on  $\Sigma_d$  respectively.  $K_{\mu\nu}$  is given as

$$K_{\mu\nu} = \frac{1}{2} \frac{\partial G_{\mu\nu}}{\partial r}, \quad K = G^{\mu\nu} K_{\mu\nu} \quad (8.3)$$

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<sup>6</sup>The very interesting attempt based on counterterm method to calculate 8-dimensional CA has been performed in ref.[40]. However, the explicit result was not obtained there. Moreover, as discussed in section 7 that counterterm method in higher dimensions may lead to some ambiguous result.

In the canonical formalism,  $\mathcal{L}_{d+1}$  is rewritten by using the canonical momenta  $\Pi_{\mu\nu}$  and Hamiltonian density  $\mathcal{H}$  as

$$\mathcal{L}_{d+1} = \Pi^{\mu\nu} \frac{\partial G_{\mu\nu}}{\partial r} + \mathcal{H}, \quad \mathcal{H} \equiv \frac{1}{d-1} (\Pi_\mu^\mu)^2 - \Pi_{\mu\nu}^2 + V - R. \quad (8.4)$$

The equation of motion for  $\Pi^{\mu\nu}$  leads to

$$\Pi^{\mu\nu} = K^{\mu\nu} - G^{\mu\nu} K. \quad (8.5)$$

The Hamilton constraint  $\mathcal{H} = 0$  leads to the Hamilton-Jacobi equation (flow equation)

$$\{S, S\}(x) = \sqrt{G} \mathcal{L}_d(x) \quad (8.6)$$

$$\{S, S\}(x) \equiv \frac{1}{\sqrt{G}} \left[ -\frac{1}{d-1} \left( G_{\mu\nu} \frac{\delta S}{\delta G_{\mu\nu}} \right)^2 + \left( \frac{\delta S}{\delta G_{\mu\nu}} \right)^2 \right], \quad (8.7)$$

$$\mathcal{L}_d(x) \equiv V - R[G]. \quad (8.8)$$

One can decompose the action  $S$  into a local and non-local part discussed in ref.[47] as follows

$$S[G(x)] = S_{loc}[G(x)] + \Gamma[G(x)], \quad (8.9)$$

Here  $S_{loc}[G(x)]$  is tree level action and  $\Gamma$  contains the higher-derivative and non-local terms. In the following discussion, we take the systematic method of ref.[66], which is weight calculation. The  $S_{loc}[G]$  can be expressed as a sum of local terms

$$S_{loc}[G(x)] = \int d^d x \sqrt{G} \mathcal{L}_{loc}(x) = \int d^d x \sqrt{G} \sum_{w=0,2,4,\dots} [\mathcal{L}_{loc}(x)]_w \quad (8.10)$$

The weight  $w$  is defined by following rules;

$$G_{\mu\nu}, \Gamma : \text{weight } 0, \quad \partial_\mu : \text{weight } 1, \quad R, R_{\mu\nu} : \text{weight } 2, \quad \frac{\delta \Gamma}{\delta G_{\mu\nu}} : \text{weight } d.$$

Using these rules and (8.6), one obtains the equations, which depend on the weight as

$$\sqrt{G} \mathcal{L}_d = [\{S_{loc}, S_{loc}\}]_0 + [\{S_{loc}, S_{loc}\}]_2 \quad (8.11)$$

$$0 = [\{S_{loc}, S_{loc}\}]_w \quad (w = 4, 6, \dots, d-2), \quad (8.12)$$

$$0 = 2[\{S_{loc}, \Gamma\}]_d + [\{S_{loc}, S_{loc}\}]_d \quad (8.13)$$

The above equations which determine  $[\mathcal{L}_{loc}]_w$ ,  $[\mathcal{L}_{loc}]_0$  and  $[\mathcal{L}_{loc}]_2$  are parametrized by

$$[\mathcal{L}_{loc}]_0 = W, \quad [\mathcal{L}_{loc}]_2 = -\Phi R. \quad (8.14)$$

Thus one can solve (8.11) as

$$V = -\frac{d}{4(d-1)} W^2, \quad -1 = \frac{d-2}{2(d-1)} W \Phi. \quad (8.15)$$

Setting  $V = 2\Lambda = -d(d-1)/l^2$ , where  $\Lambda$  is the bulk cosmological constant and the parameter  $l$  is the radius of the asymptotic  $\text{AdS}_{d+1}$ , we obtain  $W$  and  $\Phi$  as

$$W = -\frac{2(d-1)}{l}, \quad \Phi = \frac{l}{d-2}. \quad (8.16)$$

To obtain the higher weight ( $w \geq 4$ ) local terms related with CA, we introduce a local term  $[\mathcal{L}_{loc}]_4$  as follows

$$[\mathcal{L}_{loc}]_4 = XR^2 + YR_{\mu\nu}R^{\mu\nu} + ZR_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}. \quad (8.17)$$

Here  $X, Y$  and  $Z$  are some constants determined by (8.12). The calculation of  $[\{S_{loc}, S_{loc}\}]_4$  was done in [66] as

$$\begin{aligned} \frac{1}{\sqrt{G}} [\{S_{loc}, S_{loc}\}]_4 = & -\frac{W}{2(d-1)} \left( (d-4)X - \frac{dl^3}{4(d-1)(d-2)^2} \right) R^2 \\ & -\frac{W}{2(d-1)} \left( (d-4)Y + \frac{l^3}{(d-2)^2} \right) R_{\mu\nu}R^{\mu\nu} - \frac{d-4}{2(d-1)} WZR_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} \\ & + \left( 2X + \frac{d}{2(d-1)}Y + \frac{2}{d-1}Z \right). \end{aligned} \quad (8.18)$$

For  $d \geq 6$ , from  $[\{S_{loc}, S_{loc}\}]_4 = 0$  one finds

$$X = \frac{dl^3}{4(d-1)(d-2)^2(d-4)}, \quad Y = -\frac{l^3}{(d-2)^2(d-4)}, \quad Z = 0. \quad (8.19)$$

Using them, one can calculate  $[\{S_{loc}, S_{loc}\}]_6$  as [66]

$$\begin{aligned} \frac{1}{\sqrt{G}} [\{S_{loc}, S_{loc}\}]_6 = & \Phi \left[ \left( -4X + \frac{d+2}{2(d-1)}Y \right) RR_{\mu\nu}R^{\mu\nu} + \frac{d+2}{2(d-1)}XR^3 \right. \\ & -4YR^{\mu\lambda}R^{\nu\sigma}R_{\mu\nu\lambda\sigma} + (4X+2Y)R^{\mu\nu}\nabla_\mu\nabla_\nu R - 2YR^{\mu\nu}\nabla^2 R_{\mu\nu} \\ & \left. + \left( -2X - \frac{d-2}{2(d-1)}Y \right) R\nabla^2 R \right] + (\text{contributions from } [\mathcal{L}_{loc}]_6) \\ = & l^4 \left[ -\frac{3d+2}{2(d-1)(d-2)^2(d-4)}RR_{\mu\nu}R^{\mu\nu} + \frac{d(d+2)}{8(d-1)^2(d-2)^3(d-4)}R^3 \right. \\ & + \frac{4}{(d-2)^3(d-4)}R^{\mu\lambda}R^{\nu\sigma}R_{\mu\nu\lambda\sigma} - \frac{1}{(d-1)(d-2)^2(d-4)}R^{\mu\nu}\nabla_\mu\nabla_\nu R \\ & + \frac{2}{(d-2)^3(d-4)}R^{\mu\nu}\nabla^2 R_{\mu\nu} - \frac{1}{(d-1)(d-2)^3(d-4)}R\nabla^2 R \left. \right] \\ & + (\text{contributions from } [\mathcal{L}_{loc}]_6). \end{aligned} \quad (8.20)$$

The flow equation of the weight  $d$  (8.13), which is related with the CA in  $d$ -dimensions [47, 66], is written by

$$-\frac{W}{2(d-1)}\frac{1}{\sqrt{G}}G_{\mu\nu}\frac{\delta\Gamma}{\delta G_{\mu\nu}} = -[\{S_{loc}, S_{loc}\}]_d. \quad (8.21)$$



This  $G_{\mu\nu} \frac{\delta\Gamma}{\delta G_{\mu\nu}}$  can be regarded as the sum of CA  $\mathcal{W}_d$  and the total derivative term  $\nabla_\mu \mathcal{J}_d^\mu$  in  $d$ -dimensions. Thus we rewrite (8.21) as following

$$\kappa^2 \mathcal{W}_d + \nabla_\mu \mathcal{J}_d^\mu = \frac{d-1}{W\sqrt{G}} [\{S_{loc}, S_{loc}\}]_d. \quad (8.22)$$

Here  $\kappa^2$  is  $d+1$ -dimensional gravitational coupling. Using the above relation, one can get the holographic CA in 4-dimensions from (8.18):

$$\kappa^2 \mathcal{W}_4 = -\frac{l}{2\sqrt{G}} [\{S_{loc}, S_{loc}\}]_4 = l^3 \left( \frac{1}{24} R^2 - \frac{1}{8} R_{\mu\nu} R^{\mu\nu} \right). \quad (8.23)$$

This agrees with the result in [18] calculated by another method (using AdS/CFT duality). Further, the above calculation can be extended to include dilaton (a scalar). The CA in 6-dimensions is calculated from (8.20) as

$$\begin{aligned} \kappa^2 \mathcal{W}_6 &= -\frac{l}{2\sqrt{G}} [\{S_{loc}, S_{loc}\}]_6 \\ &= l^5 \left( \frac{1}{128} R R_{\mu\nu} R^{\mu\nu} - \frac{3}{3200} R^3 - \frac{1}{64} R^{\mu\lambda} R^{\nu\sigma} R_{\mu\nu\lambda\sigma} \right. \\ &\quad \left. + \frac{1}{320} R^{\mu\nu} \nabla_\mu \nabla_\nu R - \frac{1}{128} R^{\mu\nu} \nabla^2 R_{\mu\nu} + \frac{1}{1280} R \nabla^2 R \right), \end{aligned} \quad (8.24)$$

which coincides exactly with 6-dimensional CA in [18]. Above discussions have already been performed in ref.[66].

The local terms of weight 6:  $[\mathcal{L}_{loc}]_6$  is assumed to be

$$\begin{aligned} [\mathcal{L}_{loc}]_6 &= aR^3 + bR R_{\mu\nu} R^{\mu\nu} + cR R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + eR_{\mu\nu\lambda\sigma} R^{\mu\rho} R^{\nu\sigma} \\ &\quad + f \nabla_\mu R \nabla^\mu R + g \nabla_\mu R_{\nu\rho} \nabla^\mu R^{\nu\rho} + h \nabla_\mu R_{\nu\rho\sigma\tau} \nabla^\mu R^{\nu\rho\sigma\tau} + j R^{\mu\nu} R_\nu^\rho R_{\rho\mu}. \end{aligned} \quad (8.25)$$

Adding above terms to (8.20), we obtain

$$\begin{aligned} \frac{1}{\sqrt{G}} [\{S_{loc}, S_{loc}\}]_6 &= \left( b \left( \frac{d}{2} - 3 \right) \frac{2}{l} - \frac{(3d+2)l^4}{2(d-1)(d-2)^3(d-4)} \right) R R_{\mu\nu} R^{\mu\nu} \\ &\quad + \left( a \left( \frac{d}{2} - 3 \right) \frac{2}{l} + \frac{d(d+2)l^4}{8(d-1)^2(d-2)^3(d-4)} \right) R^3 \\ &\quad + \left( \left\{ -e \left( \frac{d}{2} + 2 \right) - 2g - \frac{3j}{2}(2-d) \right\} \frac{2}{l} + \frac{4l^4}{(d-2)^3(d-4)} \right) R^{\mu\lambda} R^{\nu\sigma} R_{\mu\nu\lambda\sigma} \\ &\quad + \left( \left\{ b(2-d) - 4c + e \left( \frac{d}{2} - 1 \right) + \frac{3j}{2}(2-d) \right\} \frac{2}{l} \right. \\ &\quad \left. - \frac{l^4}{(d-1)(d-2)^2(d-4)} \right) R^{\mu\nu} \nabla_\mu \nabla_\nu R \\ &\quad + \left( \{2b(1-d) - de + 2g - 3j\} \frac{2}{l} + \frac{2l^4}{(d-2)^3(d-4)} \right) R^{\mu\nu} \nabla^2 R_{\mu\nu} \\ &\quad + \left( \left\{ 6a(1-d) - b \left( 1 + \frac{d}{2} \right) - 2c - \frac{1}{2}e + 2f \right\} \frac{2}{l} \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{l^4}{(d-1)(d-2)^3(d-4)} \Big) R \nabla^2 R \\
& + \left( \frac{d}{2}g + 2h + 2f(d-1) \right) \frac{2}{l} \nabla^4 R + \left( \frac{d}{2} - 3 \right) \frac{2c}{l} R R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\
& + \left( 6a(1-d) - db - 4c - \frac{3}{4}e + \left( \frac{d}{2} - 1 \right) f - \frac{g}{2} + \frac{3}{8}(2-d)j \right) \frac{2}{l} \nabla_\mu R \nabla^\mu R \\
& + \left( 2b(1-d) + 2e(1-d) + g \left( \frac{d}{2} - 1 \right) - 8h - 3j \right) \frac{2}{l} \nabla_\kappa R_{\mu\nu} \nabla^\kappa R^{\mu\nu} \\
& + \left( (2d-3)e + 2g + 8h + \frac{3}{2}(2-d)j \right) \frac{2}{l} \nabla_\kappa R^{\mu\nu} \nabla_\nu R_\mu^\kappa \\
& + ((d-1)e + 2g - dj) \frac{2}{l} R^{\mu\nu} R_\nu^\rho R_{\mu\rho} + (2-d) \frac{2e}{l} R^{\mu\nu\rho\sigma} \nabla_\mu \nabla_\rho R_{\nu\sigma} \\
& + \left( 2c(1-d) + \left( \frac{d}{2} - 1 \right) h \right) \frac{2}{l} \nabla_\alpha R_{\mu\nu\rho\sigma} \nabla^\alpha R^{\mu\nu\rho\sigma} \\
& + (2c(1-d) + 2h) \frac{2}{l} R_{\mu\nu\rho\sigma} \nabla^2 R^{\mu\nu\rho\sigma} \\
& + \left( 4R^{\mu\rho\sigma\tau} R_{\lambda\mu} R_{\rho\sigma\tau}^\lambda - 4R^{\mu\rho\sigma\tau} R_{\mu\lambda\rho}^\nu R_{\nu\tau\sigma}^\lambda \right) \frac{2h}{l} \\
& + \left( -8R^{\mu\rho\sigma\tau} R_{\mu\lambda\sigma}^\nu R_{\tau\nu\rho}^\lambda + 4\nabla_\nu R_{\mu\rho\sigma\tau} \nabla^\mu R^{\nu\rho\sigma\tau} \right) \frac{2h}{l} \\
& = \left[ \left( \frac{(d-6)b}{l} - \frac{(3d+2)l^4}{2(d-1)(d-2)^2(d-4)} \right) R R_{\mu\nu} R^{\mu\nu} \right. \\
& + \left( \frac{(d-6)a}{l} + \frac{d(d+2)l^4}{8(d-1)^2(d-2)^3(d-4)} \right) R^3 + \left( \frac{(d-6)c}{l} \right) R R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} \\
& + \left( \frac{(d-6)e}{l} + \frac{16h}{l} + \frac{4l^4}{(d-2)^3(d-4)} \right) R^{\mu\lambda} R^{\nu\sigma} R_{\mu\nu\lambda\sigma} \\
& + \left( -\left( 3 - \frac{d}{2} \right) \frac{2f}{l} + \frac{4h}{l} + \frac{dl^4}{2(d-1)(d-2)^3(d-4)} \right) \nabla^\mu R \nabla_\mu R \\
& + \left( \frac{(d-6)g}{l} - \frac{2l^4}{(d-2)^3(d-4)} + \frac{16h}{l} \right) \nabla^\rho R^{\mu\nu} \nabla_\rho R_{\mu\nu} \\
& + \left( \frac{2j}{l} \left( \frac{d}{2} - 3 \right) - \frac{16h}{l} \right) R^{\mu\nu} R_\nu^\rho R_{\rho\mu} \\
& + \left. \left( \frac{(6-d)h}{l} \right) R_{\mu\nu\lambda\sigma} \nabla^2 R^{\mu\nu\lambda\sigma} \right] + \text{total derivative terms} . \tag{8.26}
\end{aligned}$$

For  $d \geq 8$ , from  $[\{S_{loc}, S_{loc}\}]_6 = 0$ , if one neglects the total derivative terms, the coefficients  $a, b, c, e, f, g, h, j$  are

$$\begin{aligned}
a &= -\frac{d(d+2)l^5}{8(d-1)^2(d-2)^3(d-4)(d-6)} \\
b &= \frac{(3d+2)l^5}{2(d-1)(d-2)^2(d-4)(d-6)} , \quad c = 0
\end{aligned}$$

$$\begin{aligned}
e &= -\frac{4l^5}{(d-2)^3(d-4)(d-6)}, \quad f = -\frac{dl^5}{2(d-1)(d-2)^3(d-4)(d-6)} \\
g &= \frac{2l^5}{(d-2)^3(d-4)(d-6)}, \quad h = 0, \quad j = 0.
\end{aligned} \tag{8.27}$$

We can also consider  $d = 8$  case in the same way. In  $d = 8$  case, one obtains  $\frac{1}{\sqrt{G}}[\{S_{loc}, S_{loc}\}]_8$  as follows

$$\begin{aligned}
\frac{1}{\sqrt{G}}[\{S_{loc}, S_{loc}\}]_8 &= \left( -\frac{(d+8)X^2}{4(d-1)} + \frac{(d+4)al}{2(d-1)(d-2)} \right) R^4 \\
&+ \left( 2X^2 + \frac{(-d+4)XY}{2(d-1)} - \frac{6al}{(d-2)} + \frac{(4-d)bl}{2(d-1)(d-2)} + \frac{el}{2(d-1)(d-2)} \right. \\
&\quad \left. - \frac{2fl}{(d-1)(d-2)} \right) R^2 \nabla^2 R + \left( -\frac{(d+8)Y^2}{4(d-1)} - \frac{2bl}{d-2} \right) (R^{\mu\nu} R_{\mu\nu})^2 \\
&+ \left( -4X^2 - \frac{d}{4(d-1)} Y^2 - 2XY \right) (\nabla^2 R)^2 \\
&+ \left( 4X^2 - \frac{(d+8)XY}{2(d-1)} - \frac{6al}{(d-2)} + \frac{(d+4)bl}{2(d-1)(d-2)} \right) R^2 R^{\mu\nu} R_{\mu\nu} \\
&+ (4X^2 + Y^2 + 4XY) \nabla^\mu \nabla^\nu R \nabla_\mu \nabla_\nu R \\
&+ \left( -8X^2 - 4XY + \frac{12al}{d-2} + \frac{bl}{d-1} + \frac{del}{2(d-1)(d-2)} \right) R R^{\mu\nu} \nabla_\mu \nabla_\nu R \\
&+ \left( \frac{(d-4)(-2d+1)Y^2}{2(d-1)} + 2XY - \frac{2bl}{d-2} - \frac{el}{d-2} + \frac{4fl}{d-2} \right) R^{\mu\nu} R_{\mu\nu} \nabla^2 R \\
&+ (-2Y^2 - 4XY) \nabla^2 R^{\mu\nu} \nabla_\mu \nabla_\nu R + \left( 4Y^2 - \frac{2el}{d-2} + \frac{4gl}{d-2} \right) \nabla^2 R^{\mu\nu} R_{\mu\lambda\nu\kappa} R^{\lambda\kappa} \\
&+ Y^2 \nabla^2 R_{\mu\nu} \nabla^2 R^{\mu\nu} + \left( 4Y^2 + \frac{4el}{d-2} \right) R_{\mu\kappa\nu}^\lambda R^{\mu\nu} R_{\sigma\lambda\gamma}^\kappa R^{\sigma\gamma} \\
&+ (-4Y^2 - 8XY) R_{\mu\kappa\nu}^\lambda R^{\mu\nu} \nabla_\lambda \nabla^\kappa R + \frac{4el}{d-2} R_{\kappa\lambda} R_{\sigma\mu\nu}^\kappa \nabla^\mu \nabla^\lambda R^{\nu\sigma} \\
&+ \left( 8XY - \frac{4bl}{d-2} + \frac{(-d+6)el}{2(d-1)(d-2)} + \frac{2gl}{(d-1)(d-2)} \right) R_{\lambda}^\kappa R_{\mu\kappa\nu}^\lambda R^{\mu\nu} R \\
&+ \left( 4XY - \frac{4bl}{(d-2)} - \frac{el}{(d-1)} - \frac{2gl}{(d-1)(d-2)} \right) R_{\kappa\lambda} R \nabla^2 R^{\kappa\lambda} \\
&+ \left( -\frac{6al}{d-2} - \frac{bl}{d-1} + \frac{3el}{4(d-1)(d-2)} + \frac{dfl}{2(d-1)(d-2)} \right. \\
&\quad \left. + \frac{gl}{2(d-1)(d-2)} \right) R (\nabla R)^2 + \frac{l}{d-2} (4b + 2e + 2g) R_{\kappa\lambda} R^{\mu\kappa} \nabla_\mu \nabla^\lambda R \\
&+ \frac{l}{d-2} \left( 12a + 2b + \frac{e}{2} - 2f \right) R_{\mu\nu} \nabla^\mu R \nabla^\nu R \\
&+ \frac{l}{d-2} \left( -2b - 2e + \frac{dg}{2(d-1)} \right) R \nabla_\kappa R_{\mu\nu} \nabla^\kappa R^{\mu\nu} + \left( \frac{2fl}{d-2} + \frac{gl}{2(d-1)} \right) R \nabla^4 R
\end{aligned}$$

$$\begin{aligned}
& -\frac{(4f+2g)l}{d-2}R_{\mu\nu}\nabla^\mu\nabla^\nu\nabla^2R + \frac{(4b+2e)l}{d-2}R_{\kappa\lambda}R^{\mu\nu}\nabla^\kappa\nabla^\lambda R_{\mu\nu} \\
& + \frac{l}{d-2}(4b+4e-2g)R_{\kappa\lambda}\nabla^\kappa R_{\mu\nu}\nabla^\lambda R^{\mu\nu} + \frac{l}{d-2}(4b-2g)R_{\kappa\lambda}\nabla_\mu R\nabla^\lambda R^{\mu\kappa} \\
& - \frac{l}{d-2}(4b+2e)R_{\kappa\lambda}\nabla_\mu R\nabla^\mu R^{\kappa\lambda} + \frac{l}{d-2}\left(e - \frac{2g}{d-1}\right)RR^{\mu\nu}R_{\kappa\mu}R^{\kappa\nu} \\
& + \frac{l}{(d-1)(d-2)}((2d-1)e-2g)R\nabla^\mu R^{\nu\kappa}\nabla_\nu R_{\mu\kappa} \\
& - \frac{del}{(d-1)(d-2)}RR^{\mu\nu\kappa\lambda}\nabla_\mu\nabla_\kappa R_{\nu\lambda} + \frac{2el}{d-2}R_{\mu\nu}\nabla_\kappa R^{\nu\rho}\nabla_\rho R^{\kappa\mu} \\
& - \frac{4(e+g)l}{d-2}R_{\kappa\lambda}R_{\nu\rho\mu}^\lambda R^{\kappa\rho}R^{\nu\mu} - \frac{8gl}{d-2}R_{\kappa\lambda}\nabla_\mu R^{\lambda\nu}\nabla^\mu R_\nu^\kappa - \frac{4gl}{d-2}R_{\kappa\lambda}R_\nu^\kappa\nabla^2R^{\lambda\nu} \\
& - \frac{4(e+g)l}{d-2}R_{\kappa\lambda}R_\nu^\mu\nabla_\mu\nabla^\lambda R^{\kappa\nu} - \frac{4(e-g)l}{d-2}R_{\kappa\lambda}\nabla_\mu R^{\kappa\nu}\nabla^\lambda R_\nu^\mu \\
& - \frac{(2e-4g)l}{d-2}R_\kappa^\lambda R^{\mu\nu}\nabla^2R_{\mu\lambda\nu}^\kappa + \frac{2gl}{d-2}R_{\mu\nu}\nabla^4R^{\mu\nu} \\
& + \frac{4el}{d-2}R_{\kappa\lambda}\nabla^\mu R^{\nu\sigma}\nabla^\lambda R_{\sigma\mu\nu}^\kappa - \frac{(4e-8g)l}{d-2}R^{\kappa\lambda}\nabla^\mu R_\sigma^\nu\nabla_\mu R_{\kappa\nu\lambda}^\sigma \\
& + \frac{(2e+4g)l}{d-2}R_{\kappa\lambda}R_\rho^\kappa R_\nu^\rho R^{\nu\lambda} + \frac{4el}{d-2}\left(R_{\kappa\lambda}R_\sigma^\nu R^{\rho\sigma\lambda\mu}R_{\rho\mu\nu}^\kappa + R_{\kappa\lambda}R_\nu^\sigma R^{\rho\nu\lambda\mu}R_{\sigma\mu\rho}^\kappa\right). \tag{8.28}
\end{aligned}$$

Substituting  $X, Y, a, b, e, f, g, j$  in (8.19) and (8.27) into the above equation and putting  $d = 8$ , we obtain the explicit form of  $[\{S_{loc}, S_{loc}\}]_8$  and CA in 8-dimensions

$$\begin{aligned}
-\frac{2}{l^7}\kappa^2\mathcal{W}_8 &= \frac{1}{l^6\sqrt{G}}[\{S_{loc}, S_{loc}\}]_8 \\
&= \frac{13}{889056}R^4 - \frac{1}{2352}R^2\nabla^2R - \frac{79}{36288}(R^{\mu\nu}R_{\mu\nu})^2 - \frac{1}{508032}(\nabla^2R)^2 \\
&+ \frac{53}{63504}R^2R^{\mu\nu}R_{\mu\nu} + \frac{1}{112896}\nabla^\mu\nabla^\nu R\nabla_\mu\nabla_\nu R + \frac{61}{63504}RR^{\mu\nu}\nabla_\mu\nabla_\nu R \\
&- \frac{23}{10368}R^{\mu\nu}R_{\mu\nu}\nabla^2R - \frac{1}{24192}\nabla^2R^{\mu\nu}\nabla_\mu\nabla_\nu R + \frac{1}{576}\nabla^2R^{\mu\nu}R_{\mu\lambda\nu\kappa}R^{\lambda\kappa} \\
&+ \frac{1}{20736}\nabla^2R_{\mu\nu}\nabla^2R^{\mu\nu} - \frac{7}{5184}R_{\mu\kappa\nu}^\lambda R^{\mu\nu}R_{\sigma\lambda\gamma}^\kappa R^{\sigma\gamma} - \frac{1}{12096}R_{\mu\kappa\nu}^\lambda R^{\mu\nu}\nabla_\lambda\nabla^\kappa R \\
&- \frac{1}{648}R_{\kappa\lambda}R_{\sigma\mu\nu}^\kappa\nabla^\mu\nabla^\lambda R^{\nu\sigma} - \frac{13}{3024}R_\lambda^\kappa R_{\mu\kappa\nu}^\lambda R^{\mu\nu}R - \frac{37}{9072}R_{\kappa\lambda}R\nabla^2R^{\kappa\lambda} \\
&- \frac{31}{28224}R(\nabla R)^2 + \frac{71}{18144}R_{\kappa\lambda}R^{\mu\kappa}\nabla_\mu\nabla^\lambda R + \frac{65}{28224}R_{\mu\nu}\nabla^\mu R\nabla^\nu R \\
&- \frac{23}{18144}R\nabla_\kappa R_{\mu\nu}\nabla^\kappa R^{\mu\nu} - \frac{1}{72576}R\nabla^4R - \frac{1}{6048}R_{\mu\nu}\nabla^\mu\nabla^\nu\nabla^2R \\
&+ \frac{2}{567}R_{\kappa\lambda}R^{\mu\nu}\nabla^\kappa\nabla^\lambda R_{\mu\nu} + \frac{43}{18144}R_{\kappa\lambda}\nabla^\kappa R_{\mu\nu}\nabla^\lambda R^{\mu\nu} \\
&+ \frac{71}{18144}R_{\kappa\lambda}\nabla_\mu R\nabla^\lambda R^{\mu\kappa} - \frac{2}{567}R_{\kappa\lambda}\nabla_\mu R\nabla^\mu R^{\kappa\lambda} - \frac{1}{2268}RR^{\mu\nu}R_{\kappa\mu}R^{\kappa\nu} \\
&- \frac{1}{1134}R\nabla^\mu R^{\nu\kappa}\nabla_\nu R_{\mu\kappa} + \frac{1}{2268}RR^{\mu\nu\kappa\lambda}\nabla_\mu\nabla_\kappa R_{\nu\lambda}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{1296} R_{\mu\nu} \nabla_\kappa R^{\nu\rho} \nabla_\rho R^{\kappa\mu} + \frac{1}{1296} R_{\kappa\lambda} R_{\nu\rho\mu}^\lambda R^{\kappa\rho} R^{\nu\mu} \\
& -\frac{1}{648} R_{\kappa\lambda} \nabla_\mu R^{\lambda\nu} \nabla^\mu R_\nu^\kappa - \frac{1}{1296} R_{\kappa\lambda} R_\nu^\kappa \nabla^2 R^{\lambda\nu} + \frac{1}{1296} R_{\kappa\lambda} R_\nu^\mu \nabla_\mu \nabla^\lambda R^{\kappa\nu} \\
& + \frac{1}{432} R_{\kappa\lambda} \nabla_\mu R^{\kappa\nu} \nabla^\lambda R_\nu^\mu + \frac{1}{648} R_\kappa^\lambda R^{\mu\nu} \nabla^2 R_{\mu\lambda\nu}^\kappa + \frac{1}{2592} R_{\mu\nu} \nabla^4 R^{\mu\nu} \\
& -\frac{1}{648} R_{\kappa\lambda} \nabla^\mu R^{\nu\sigma} \nabla^\lambda R_{\sigma\mu\nu}^\kappa + \frac{1}{324} R^{\kappa\lambda} \nabla^\mu R_\sigma^\nu \nabla_\mu R_{\kappa\nu\lambda}^\sigma \\
& -\frac{1}{648} \left( R_{\kappa\lambda} R_\sigma^\nu R^{\rho\sigma\lambda\mu} R_{\rho\mu\nu}^\kappa + R_{\kappa\lambda} R_\nu^\sigma R^{\rho\nu\lambda\mu} R_{\sigma\mu\rho}^\kappa \right). \tag{8.29}
\end{aligned}$$

As one can see already in 8-dimensions (and omitting total derivative terms) the explicit result for holographic CA is quite complicated. It is clear that going to higher dimensions it is getting much more complicated.

As an example, we consider de Sitter space, where curvatures are covariantly constant and given by

$$R_{\mu\nu\rho\sigma} = \frac{1}{l^2} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \ , \quad R_{\mu\nu} = \frac{d-1}{l^2} g_{\mu\nu} \ , \quad R = \frac{d(d-1)}{l^2} \ . \tag{8.30}$$

Here  $l$  is the radius of the de Sitter space and it is related to the cosmological constant  $\Lambda$  by  $\Lambda = \frac{(d-2)(d-1)}{l^2}$ . By putting  $d = 8$  in (8.30) and substituting the curvatures into (8.29), we find an expression for the anomaly:

$$\kappa^2 \mathcal{W}_8 = -\frac{l}{2\sqrt{G}} [\{S_{loc}, S_{loc}\}]_8 = -\frac{62069}{1296l} \ . \tag{8.31}$$

We should note that  $\frac{1}{\kappa^2}$  is 9-dimensional one here, then  $\kappa^2$  has the dimension of 7th power of the length.

In refs.[67, 68] the QFT conformal anomalies coming from scalar and spinor fields in 8-dimensional de Sitter space are found

$$T_{\text{scalar}} = -\frac{23}{34560\pi^4 l^8} \ , \quad T_{\text{spinor}} = -\frac{2497}{34560\pi^4 l^8} \ . \tag{8.32}$$

If there is supersymmetry, the number of the scalars is related with that of the spinors. For example, consider the matter supermultiplet and take only scalar-spinor part of it (one real scalar and one Dirac spinor) as vector is not conformally invariant in 8-dimensions. If there is  $N^4$  pairs of scalars and spinors, the total anomaly should be given by

$$\mathcal{W}_8 = N^4 (T_{\text{scalar}} + T_{\text{spinor}}) = -\frac{7N^4}{6(2\pi)^4 l^8} \ . \tag{8.33}$$

By comparing (8.33) with (8.31), we find

$$\frac{1}{\kappa^2} = \frac{216N^4}{8867(2\pi)^4 l^7} \ , \tag{8.34}$$

which might be useful to establish the proposal for  $\text{AdS}_9/\text{CFT}_8$ .

Of course, the above relation gives only the indication (the numerical factor is definitely wrong) as we considered only scalar-spinor part of non-conformal multiplet. On the same time it is known that for  $\text{AdS}_7/\text{CFT}_6$  correspondence the tensor multiplet gives brane CFT while for 4-dimensional the gauge fields play the important role (super Yang-Mills theory). As far as we know the rigorous proposal for 8-dimensional brane CFT does not exist yet. However, it is evident that not only scalars and spinors but also other fields will be part of 8-dimensional CFT. It would be extremely interesting to construct the candidate for such theory. Then the above 8-dimensional holographic anomaly may be used to check the correctness of such proposal.

## 9 Summary

In this section, we summarize the results. First, we calculated CA of boundary CFT in 2 and 4-dimensions with broken conformal invariance by using AdS/CFT correspondence. We derived such CA from the bosonic part of gauged SUGRA including single scalar with arbitrary scalar potential in 3 and 5-dimensions. Within holographic RG where identification of dilaton with some coupling constant is made, we proposed the candidate c-function for 2 and 4-dimensional boundary QFT from holographic CA. Having examined some examples of scalar potentials, we checked the c-theorem and compared this c-function with the other proposals for it. It is shown that such proposal gives monotonic and positive c-function for few examples of dilatonic potential.

Next, we constructed surface counterterm for gauged SUGRA with single scalar and arbitrary scalar potential in 3 and 5-dimensions. As a result, the finite gravitational action and consistent stress tensor in asymptotically AdS space is found. Using this action, the regularized expressions for free energy, entropy and mass are derived for 5-dimensional dilatonic AdS black hole. It might be interesting to consider the calculation of surface counterterm in 5-dimensional gauged SUGRA with many scalars which is slightly easier task. However, again the application of surface counterterm for the derivation of regularized thermodynamical quantities in multi-scalar AdS black holes might be complicated.

CA from 3 and 5-dimensional gauged SUGRAs with maximally SUSY are also obtained. It corresponds to the one of dual CFT living on the boundary of asymptotically AdS space. The only condition is that parametrization of scalar coset is done so that kinetic term for scalars has the standard field theory form. The bulk potential is arbitrary subject to consistent parametrization. From the point of view of AdS/CFT correspondence the exponent of scalar should correspond to gauge coupling constant. Hence, this expression represents the (exact) CA with radiative corrections for dual CFT. We derived c-functions in UV limit by the same manner of single scalar case. Those c-function do not depend on number of scalars, which is the same result of single scalar case.

Next, we considered scheme dependence of CA calculations in case of non-trivial bulk potential and non-constant dilaton. Comparing the different formalism of calculations, one is based on de Boer-Verlinde-Verlinde (Hamilton-Jacobi formalism) and another is based on Henningson-Skenderis formalism, we found the disagreement of them. Technically, this disagreement might occur since we expand dilatonic potential in the power series on  $\rho$  and this is the reason of ambiguity and scheme dependence of holographic CA.

This result means that holographic CA including non-constant bulk potential is scheme dependent. AdS/CFT correspondence says that such holographic CA should correspond to (multi-loop) QFT CA (dilaton plays the role of coupling constants). However, QFT multi-loop CA depends on regularization (as beta-functions are also scheme dependent). Hence, scheme dependence of holographic CA is consistent with QFT expectations. That also means that two different formalisms we discussed in this thesis actually should correspond to different regularizations of dual QFT.

As we know that the rigorous proposal for 8-dimensional CFT does not exist yet, we tried to calculate 8-dimensional CA from 9-dimensional pure SUGRA by Hamilton-Jacobi formalism. To check the validity, we applied the result to de Sitter space. Comparing this CA with the known CA coming from scalar and spinor fields in 8-dimensional de Sitter space, we gave the indication of 9-dimensional gravitational constant which might be useful to establish the proposal of  $\text{AdS}_9/\text{CFT}_8$ . We expect not only scalar-spinor but also other fields of 8-dimensional CFT will have such correspondence. It would be extremely interesting to construct the candidate for such theory.

In this thesis, by using AdS/CFT duality, we considered the various aspects of CA from SUGRA including scalars with potential. We expect that these results may be very useful in explicit identification of SUGRA description (special RG flow) with the particular boundary gauge theory (or its phase) which is a very non-trivial task in AdS/CFT correspondence. It might be an interesting problem to generalize CA for including other background fields (antisymmetric tensors, gauge fields, ...).

## 10 Acknowledgments

The author thanks Prof. Akio Sugamoto for useful discussions and advice. The author would like to express gratitude to Prof. Shin'ichi Nojiri and Prof. Sergei.D. Odintsov for valuable discussions and fruitful collaborations which this thesis is based upon. The author is pleased to thank all members of Sugamoto group for their encouragement and advice. The author thanks the members of Yukawa Institute for Theoretical Physics for their useful discussions during her stay there. This work is supported in part by Japan Society for the Promotion of Science.

# Appendix

## A Coefficients of Conformal Anomaly

In this appendix, we give the explicit values of the coefficients appeared in the calculation of 4-dimensional CA in section 2.

Substituting (2.16) into (2.15), we obtain

$$g_{(1)ij} = \tilde{c}_1 R_{ij} + \tilde{c}_2 g_{ij} R + \tilde{c}_3 g_{ij} g^{kl} \partial_k \phi \partial_l \phi + \tilde{c}_4 g_{ij} \frac{\partial_k}{\sqrt{-g}} (\sqrt{-g} g^{kl} \partial_l \phi) + \tilde{c}_5 \partial_i \phi \partial_j \phi \quad (\text{A.1})$$

$$\begin{aligned} \tilde{c}_1 &= -\frac{3}{6 + \Phi} \\ \tilde{c}_2 &= -\frac{3 \{ \Phi'^2 - 6 (\Phi'' + 8 V) \}}{2 (6 + \Phi) \{ -2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V) \}} \\ \tilde{c}_3 &= \frac{-3 \Phi'^2 V + 18 V (\Phi'' + 8 V) - 2 (6 + \Phi) \Phi' V'}{2 (6 + \Phi) (-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V))} \\ \tilde{c}_4 &= -\frac{2 \Phi' V}{-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V)} \\ \tilde{c}_5 &= -\frac{V}{2 + \frac{\Phi}{3}}. \end{aligned} \quad (\text{A.2})$$

Further, substituting (2.16) and (A.1) into (2.18), we obtain

$$\begin{aligned} \phi_{(2)} &= d_1 R^2 + d_2 R_{ij} R^{ij} + d_3 R^{ij} \partial_i \phi \partial_j \phi \\ &\quad + d_4 R g^{ij} \partial_i \phi \partial_j \phi + d_5 R \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \\ &\quad + d_6 (g^{ij} \partial_i \phi \partial_j \phi)^2 + d_7 \left( \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right)^2 \\ &\quad + d_8 g^{kl} \partial_k \phi \partial_l \phi \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} d_1 &= - \left[ 9 \Phi' \left\{ 2 (12 + \Phi) \Phi'^4 - (-864 + 36 \Phi + 24 \Phi^2 + \Phi^3) \Phi''^2 \right. \right. \\ &\quad + 192 (12 + \Phi)^2 \Phi'' V + 64 (2592 + 612 \Phi + 48 \Phi^2 + \Phi^3) V^2 \\ &\quad - 2 \Phi'^2 \left( (216 + 30 \Phi + \Phi^2) \Phi'' + 144 (10 + \Phi) V \right) \\ &\quad \left. \left. + (6 + \Phi)^2 (24 + \Phi) \Phi' (\Phi''' + 8 V') \right\} \right] / \\ &\quad \left[ 8 (6 + \Phi)^2 \left\{ -2 \Phi'^2 + (24 + \Phi) \Phi'' \right\} \right. \\ &\quad \left. \times \left\{ -2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V) \right\}^2 \right] \\ d_2 &= \frac{9 (12 + \Phi) \Phi'}{4 (6 + \Phi)^2 \left\{ -2 \Phi'^2 + (24 + \Phi) \Phi'' \right\}} \end{aligned}$$



$$\begin{aligned}
d_3 &= \frac{3 (3 (12 + \Phi) \Phi' V - 2 (144 + 30 \Phi + \Phi^2) V')}{2 (6 + \Phi)^2 (-2 \Phi'^2 + (24 + \Phi) \Phi'')} \\
d_4 &= (3 (-6 (12 + \Phi) \Phi'^5 V + 6 (108 + 24 \Phi + \Phi^2) \Phi'^4 V' \\
&\quad + 4 (2592 + 684 \Phi + 48 \Phi^2 + \Phi^3) (\Phi'' + 8 V) ((9 + \Phi) \Phi'' \\
&\quad + 4 (12 + \Phi) V) V' - (6 + \Phi) \Phi'^2 (3 (144 + 30 \Phi + \Phi^2) \Phi''' V \\
&\quad + (1980 \Phi'' + 216 \Phi \Phi'' + 5 \Phi^2 \Phi'' + 27360 V + 4176 \Phi V \\
&\quad + 128 \Phi^2 V) V') + 2 \Phi'^3 (3 (216 + 30 \Phi + \Phi^2) \Phi'' V \\
&\quad - 2 (-2160 V^2 - 216 \Phi V^2 + 864 V'' + 324 \Phi V'' \\
&\quad + 36 \Phi^2 V'' + \Phi^3 V'')) + \Phi' (3 (-864 + 36 \Phi + 24 \Phi^2 + \Phi^3) \Phi''^2 V \\
&\quad + 2 \Phi'' (-41472 V^2 - 6912 \Phi V^2 - 288 \Phi^2 V^2 + 15552 V'' \\
&\quad + 6696 \Phi V'' + 972 \Phi^2 V'' + 54 \Phi^3 V'' + \Phi^4 V'')) \\
&\quad - 2 (248832 V^3 + 58752 \Phi V^3 + 4608 \Phi^2 V^3 \\
&\quad + 96 \Phi^3 V^3 + 15552 \Phi''' V' + 6696 \Phi \Phi''' V' + 972 \Phi^2 \Phi''' V' \\
&\quad + 54 \Phi^3 \Phi''' V' + \Phi^4 \Phi''' V' + 124416 V'^2 + 53568 \Phi V'^2 \\
&\quad + 7776 \Phi^2 V'^2 + 432 \Phi^3 V'^2 + 8 \Phi^4 V'^2 - 124416 V V'' \\
&\quad - 53568 \Phi V V'' - 7776 \Phi^2 V V'' - 432 \Phi^3 V V'' \\
&\quad - 8 \Phi^4 V V'')))))/ \\
&\quad (4 (6 + \Phi)^2 (-2 \Phi'^2 + (24 + \Phi) \Phi'') (-2 \Phi'^2 \\
&\quad + (18 + \Phi) (\Phi'' + 8 V))^2) \\
d_5 &= -(3 (2 \Phi'^4 V + 2 (432 + 42 \Phi + \Phi^2) \Phi'' V (\Phi'' + 8 V) \\
&\quad + \Phi'^2 V ((6 + \Phi) \Phi'' - 8 (162 + 7 \Phi) V) - 4 (24 + \Phi) \Phi'^3 V' \\
&\quad - 2 (432 + 42 \Phi + \Phi^2) \Phi' (\Phi''' V - \Phi'' V')))/ \\
&\quad (2 (2 \Phi'^2 - (24 + \Phi) \Phi'') (-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V))^2) \\
d_6 &= -(-54 (12 + \Phi) \Phi'^5 V^2 + 12 (828 + 168 \Phi + 5 \Phi^2) \Phi'^4 V V' \\
&\quad + 4 (2592 + 684 \Phi + 48 \Phi^2 + \Phi^3) \\
&\quad V' (54 \Phi''^2 V + 4608 V^3 + 192 \Phi V^3 + 108 \Phi''' V' + 24 \Phi \Phi''' V' \\
&\quad + \Phi^2 \Phi''' V' + 864 V'^2 + 192 \Phi V'^2 + 8 \Phi^2 V'^2 - 1728 V V'' \\
&\quad - 384 \Phi V V'' - 16 \Phi^2 V V'' + 2 \Phi'' (504 V^2 + 12 \Phi V^2 - 108 V'' \\
&\quad - 24 \Phi V'' - \Phi^2 V'')) + (6 + \Phi) \Phi'^2 (9 (144 + 30 \Phi + \Phi^2) \Phi''' V^2 \\
&\quad - 2 V' (14796 \Phi'' V + 1368 \Phi \Phi'' V + 33 \Phi^2 \Phi'' V \\
&\quad + 88992 V^2 + 4680 \Phi V^2 + 36 \Phi^2 V^2 - 20736 V'' \\
&\quad - 5472 \Phi V'' - 384 \Phi^2 V'' - 8 \Phi^3 V'')) \\
&\quad + 2 \Phi'^3 (27 (216 + 30 \Phi + \Phi^2) \Phi'' V^2 + 4 (12312 V^3 \\
&\quad + 1836 \Phi V^3 + 72 \Phi^2 V^3 + 2376 V'^2 + 864 \Phi V'^2 + 90 \Phi^2 V'^2 \\
&\quad + 2 \Phi^3 V'^2 + 2592 V V'' + 972 \Phi V V'' + 108 \Phi^2 V V'' \\
&\quad + 3 \Phi^3 V V'')) - \Phi' (27 (2304 + 516 \Phi + 40 \Phi^2 + \Phi^3) \Phi''^2 V^2 \\
&\quad + 4 \Phi'' (217728 V^3 + 44064 \Phi V^3 + 3024 \Phi^2 V^3 + 72 \Phi^3 V^3
\end{aligned}$$

$$\begin{aligned}
& +81648 V'^2 + 34992 \Phi V'^2 + 5040 \Phi^2 V'^2 + 276 \Phi^3 V'^2 \\
& +5 \Phi^4 V'^2 + 46656 V V'' + 20088 \Phi V V'' \\
& +2916 \Phi^2 V V'' + 162 \Phi^3 V V'' + 3 \Phi^4 V V'') \\
& +4 V (746496 V^3 + 129600 \Phi V^3 + 6912 \Phi^2 V^3 \\
& +144 \Phi^3 V^3 - 46656 \Phi''' V' - 20088 \Phi \Phi''' V' - 2916 \Phi^2 \Phi''' V' \\
& -162 \Phi^3 \Phi''' V' - 3 \Phi^4 \Phi''' V' - 404352 V'^2 - 177984 \Phi V'^2 \\
& -26784 \Phi^2 V'^2 - 1584 \Phi^3 V'^2 - 32 \Phi^4 V'^2 + 373248 V V'' \\
& +160704 \Phi V V'' + 23328 \Phi^2 V V'' + 1296 \Phi^3 V V'' \\
& +24 \Phi^4 V V''))/(8 (6 + \Phi)^2 (-2 \Phi'^2 + (24 + \Phi) \Phi'') (-2 \Phi'^2 \\
& +(18 + \Phi) (\Phi'' + 8 V))^2) \\
d_7 = & (2 V (36 \Phi'^3 V - 3 (18 + \Phi) \Phi' V \\
& ((26 + \Phi) \Phi'' - 8 (18 + \Phi) V) + 4 (432 + 42 \Phi + \Phi^2) \Phi'^2 V' \\
& +(18 + \Phi)^2 (24 + \Phi) (\Phi''' V - 2 (\Phi'' + 4 V) V')))/ \\
& ((2 \Phi'^2 - (24 + \Phi) \Phi'') (-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V))^2) \\
d_8 = & -(6 \Phi'^4 V^2 - 4 (156 + 5 \Phi) \Phi'^3 V V' - 2 (18 + \Phi) \Phi' V \\
& (3 (24 + \Phi) \Phi''' V + (-276 \Phi'' - 11 \Phi \Phi'' + 480 V + 32 \Phi V) V') \\
& +2 (432 + 42 \Phi + \Phi^2) (3 \Phi''^2 V^2 \\
& +2 (18 + \Phi) V (-\Phi''' V' + 8 V V'')) \\
& +2 \Phi'' (12 V^3 + 18 V'^2 + \Phi V'^2 + 18 V V'' + \Phi V V'')) \\
& +\Phi'^2 (3 (6 + \Phi) \Phi'' V^2 - 8 (486 V^3 + 21 \Phi V^3 + 432 V'^2 \\
& +42 \Phi V'^2 + \Phi^2 V'^2 + 432 V V'' + 42 \Phi V V'' + \Phi^2 V V'')))/ \\
& (2 (2 \Phi'^2 - (24 + \Phi) \Phi'') (-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V))^2) .
\end{aligned}$$

Substituting (2.16), (A.1) and (A.3) into (2.17), one gets

$$\begin{aligned}
g^{ij}g_{(2)ij} = & f_1 R^2 + f_2 R_{ij} R^{ij} + f_3 R^{ij} \partial_i \phi \partial_j \phi \\
& + f_4 R g^{ij} \partial_i \phi \partial_j \phi + f_5 R \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \\
& + f_6 (g^{ij} \partial_i \phi \partial_j \phi)^2 + f_7 \left( \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right)^2 \\
& + f_8 g^{kl} \partial_k \phi \partial_l \phi \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \tag{A.4} \\
f_1 = & \left[ 9 \left\{ 2 \Phi'^6 - 72 (12 + \Phi) \Phi'' (\Phi'' + 8 V)^2 \right. \right. \\
& - 2 \Phi'^4 ((24 + \Phi) \Phi'' + 8 (18 + \Phi) V) \\
& + \Phi'^2 ((324 + 12 \Phi - \Phi^2) \Phi''^2 \\
& + 8 (540 + 48 \Phi + \Phi^2) \Phi'' V + 64 (180 + 24 \Phi + \Phi^2) V^2) \\
& \left. \left. + (6 + \Phi)^2 \Phi'^3 (\Phi''' + 8 V') \right\} \right] / \\
& \left[ 2 (6 + \Phi)^2 \left\{ -2 \Phi'^2 + (24 + \Phi) \Phi'' \right\} \right]
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ -2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V) \right\}^2 \Big] \\
f_2 &= - \frac{9 (\Phi'^2 - 6 \Phi'')}{(6 + \Phi)^2 \{-2 \Phi'^2 + (24 + \Phi) \Phi''\}} \\
f_3 &= \frac{6 (-3 \Phi'^2 V + 18 \Phi'' V + 2 (6 + \Phi) \Phi' V')}{(6 + \Phi)^2 (-2 \Phi'^2 + (24 + \Phi) \Phi'')} \\
f_4 &= -(3 (-12 \Phi'^6 V + 432 (12 + \Phi) \Phi'' V (\Phi'' + 8 V)^2 \\
& + 8 (6 + \Phi) \Phi'^5 V' + (6 + \Phi) \Phi' ((1044 + 168 \Phi + 7 \Phi^2) \Phi''^2 \\
& + 8 (1476 + 192 \Phi + 7 \Phi^2) \Phi'' V \\
& + 256 (216 + 30 \Phi + \Phi^2) V^2) V' \\
& - 2 (6 + \Phi) \Phi'^3 (3 (6 + \Phi) \Phi''' V \\
& + (66 \Phi'' + 3 \Phi \Phi'' + 912 V + 88 \Phi V) V') \\
& + 4 \Phi'^4 (3 (24 + \Phi) \Phi'' V - 2 (-216 V^2 - 12 \Phi V^2 + 36 V'' \\
& + 12 \Phi V'' + \Phi^2 V'')) + 2 \Phi'^2 (3 (-324 - 12 \Phi + \Phi^2) \Phi''^2 V \\
& + 2 (18 + \Phi) \Phi'' (-360 V^2 - 12 \Phi V^2 + 36 V'' + 12 \Phi V'' + \Phi^2 V'')) \\
& - 2 (17280 V^3 + 2304 \Phi V^3 \\
& + 96 \Phi^2 V^3 + 648 \Phi''' V' + 252 \Phi \Phi''' V' + 30 \Phi^2 \Phi''' V' \\
& + \Phi^3 \Phi''' V' + 5184 V'^2 + 2016 \Phi V'^2 + 240 \Phi^2 V'^2 + 8 \Phi^3 V'^2 \\
& - 5184 V V'' - 2016 \Phi V V'' - 240 \Phi^2 V V'' - 8 \Phi^3 V V'')))/ \\
& (2 (6 + \Phi)^2 (-2 \Phi'^2 + (24 + \Phi) \Phi'') (-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V))^2) \\
f_5 &= -(3 \Phi' (\Phi'' V (-3 (10 + \Phi) \Phi'' + 8 (42 + \Phi) V) \\
& + \Phi'^2 (-6 \Phi'' V + 32 V^2) + 8 \Phi'^3 V' \\
& + 4 (18 + \Phi) \Phi' (\Phi''' V - \Phi'' V')))/ \\
& ((2 \Phi'^2 - (24 + \Phi) \Phi'') (-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V))^2) \\
f_6 &= (-54 \Phi'^6 V^2 + 72 (6 + \Phi) \Phi'^5 V V' \\
& + 2 \Phi'' (54 (252 + 30 \Phi + \Phi^2) \Phi''^2 V^2 \\
& + 24 V (36288 V^3 + 4320 \Phi V^3 + 144 \Phi^2 V^3 \\
& + 11664 V'^2 + 5184 \Phi V'^2 + 792 \Phi^2 V'^2 + 48 \Phi^3 V'^2 + \Phi^4 V'^2) \\
& + \Phi'' (217728 V^3 + 25920 \Phi V^3 + 864 \Phi^2 V^3 \\
& + 11664 V'^2 + 5184 \Phi V'^2 + 792 \Phi^2 V'^2 + 48 \Phi^3 V'^2 + \Phi^4 V'^2)) \\
& + (6 + \Phi) \Phi'^3 (9 (6 + \Phi) \Phi''' V^2 - 2 V' (666 \Phi'' V + 39 \Phi \Phi'' V \\
& + 4392 V^2 + 156 \Phi V^2 - 864 V'' - 192 \Phi V'' - 8 \Phi^2 V'')) \\
& + (6 + \Phi) \Phi' V' (3 (1548 + 120 \Phi + \Phi^2) \Phi''^2 V \\
& + 8 \Phi'' (11124 V^2 + 1152 \Phi V^2 \\
& + 27 \Phi^2 V^2 - 1944 V'' - 540 \Phi V'' \\
& - 42 \Phi^2 V'' - \Phi^3 V'')) + 4 (18 + \Phi) \\
& (4608 V^3 + 192 \Phi V^3 + 108 \Phi''' V' \\
& + 24 \Phi \Phi''' V' + \Phi^2 \Phi''' V' + 864 V'^2)
\end{aligned}$$

$$\begin{aligned}
& +192 \Phi V'^2 + 8 \Phi^2 V'^2 - 1728 V V'' - 384 \Phi V V'' - 16 \Phi^2 V V'')) \\
& +6 \Phi'^4 (9 (24 + \Phi) \Phi'' V^2 + 4 (324 V^3 + 18 \Phi V^3 \\
& +36 V'^2 + 12 \Phi V'^2 + \Phi^2 V'^2 + 36 V V'' + 12 \Phi V V'' + \Phi^2 V V'')) \\
& -\Phi'^2 (27 (396 + 36 \Phi + \Phi^2) \Phi''^2 V^2 + 4 \Phi'' (29160 V^3 \\
& +2592 \Phi V^3 + 54 \Phi^2 V^3 + 4104 V'^2 + 1620 \Phi V'^2 + 198 \Phi^2 V'^2 \\
& +7 \Phi^3 V'^2 + 1944 V V'' + 756 \Phi V V'' + 90 \Phi^2 V V'' + 3 \Phi^3 V V'')) \\
& +4 V (67392 V^3 + 6912 \Phi V^3 + 144 \Phi^2 V^3 \\
& -1944 \Phi''' V' - 756 \Phi \Phi''' V' \\
& -90 \Phi^2 \Phi''' V' - 3 \Phi^3 \Phi''' V' - 5184 V'^2 - 2016 \Phi V'^2 - 240 \Phi^2 V'^2 \\
& -8 \Phi^3 V'^2 + 15552 V V'' + 6048 \Phi V V'' \\
& +720 \Phi^2 V V'' + 24 \Phi^3 V V'')))/ \\
& (2 (6 + \Phi)^2 (-2 \Phi'^2 + (24 + \Phi) \Phi'') (-2 \Phi'^2 \\
& +(18 + \Phi) (\Phi'' + 8 V))^2) \\
f_7 = & -(4 V (4 \Phi'^4 V - 2 (78 + 5 \Phi) \Phi'^2 \Phi'' V \\
& +(18 + \Phi)^2 \Phi'' V (\Phi'' + 24 V) \\
& +8 (18 + \Phi) \Phi'^3 V' + 2 (18 + \Phi)^2 \Phi' (\Phi''' V - 2 (\Phi'' + 4 V) V')))/ \\
& ((2 \Phi'^2 - (24 + \Phi) \Phi'') (-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V))^2) \\
f_8 = & (-56 \Phi'^4 V V' - 4 (18 + \Phi)^2 \Phi'' V (\Phi'' + 24 V) V' \\
& -4 \Phi'^2 V (3 (18 + \Phi) \Phi''' V \\
& -(246 \Phi'' + 15 \Phi \Phi'' + 288 V + 16 \Phi V) V') \\
& +2 \Phi'^3 (9 \Phi'' V^2 - 8 (6 V^3 + 18 V'^2 + \Phi V'^2 \\
& +18 V V'' + \Phi V V'')) + \Phi' \\
& (9 (10 + \Phi) \Phi''^2 V^2 \\
& +8 (18 + \Phi)^2 V (-\Phi''' V' + 8 V V'') - 8 \Phi'' (126 V^3 + 3 \Phi V^3 \\
& -324 V'^2 - 36 \Phi V'^2 - \Phi^2 V'^2 \\
& -324 V V'' - 36 \Phi V V'' - \Phi^2 V V'')))/ \\
& ((2 \Phi'^2 - (24 + \Phi) \Phi'') (-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V))^2) .
\end{aligned}$$

Finally substituting (2.16), (A.1), (A.3) and (A.4) into the expression for the anomaly (2.14), we obtain,

$$\begin{aligned}
T = & -\frac{1}{8\pi G} \left[ h_1 R^2 + h_2 R_{ij} R^{ij} + h_3 R^{ij} \partial_i \phi \partial_j \phi \right. \\
& + h_4 R g^{ij} \partial_i \phi \partial_j \phi + h_5 R \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \\
& + h_6 (g^{ij} \partial_i \phi \partial_j \phi)^2 + h_7 \left( \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right)^2 \\
& \left. + h_8 g^{kl} \partial_k \phi \partial_l \phi \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right] \tag{A.5}
\end{aligned}$$

$$\begin{aligned}
h_1 &= \left[ 3 \left\{ (24 - 10 \Phi) \Phi'^6 \right. \right. \\
&\quad + (62208 + 22464 \Phi + 2196 \Phi^2 + 72 \Phi^3 + \Phi^4) \Phi'' (\Phi'' + 8 V)^2 \\
&\quad + 2 \Phi'^4 \left\{ (108 + 162 \Phi + 7 \Phi^2) \Phi'' + 72 (-8 + 14 \Phi + \Phi^2) V \right\} \\
&\quad - 2 \Phi'^2 \left\{ (6912 + 2736 \Phi + 192 \Phi^2 + \Phi^3) \Phi''^2 \right. \\
&\quad + 4 (11232 + 6156 \Phi + 552 \Phi^2 + 13 \Phi^3) \Phi'' V \\
&\quad + 32 (-2592 + 468 \Phi + 96 \Phi^2 + 5 \Phi^3) V^2 \left. \right\} \\
&\quad \left. - 3 (-24 + \Phi) (6 + \Phi)^2 \Phi'^3 (\Phi''' + 8 V') \right\} \Big] / \\
&\quad \left[ 16 (6 + \Phi)^2 \left\{ -2 \Phi'^2 + (24 + \Phi) \Phi'' \right\} \left\{ -2 \Phi'^2 \right. \right. \\
&\quad \left. \left. + (18 + \Phi) (\Phi'' + 8 V) \right\}^2 \right] \\
h_2 &= -\frac{3 \left\{ (12 - 5 \Phi) \Phi'^2 + (288 + 72 \Phi + \Phi^2) \Phi'' \right\}}{8 (6 + \Phi)^2 \left\{ -2 \Phi'^2 + (24 + \Phi) \Phi'' \right\}} \\
h_3 &= -(3 ((12 - 5 \Phi) \Phi'^2 V + (288 + 72 \Phi + \Phi^2) \Phi'' V \\
&\quad + 2 (-144 - 18 \Phi + \Phi^2) \Phi' V')) / \\
&\quad (4 (6 + \Phi)^2 (-2 \Phi'^2 + (24 + \Phi) \Phi'')) \\
h_4 &= (-6 (-12 + 5 \Phi) \Phi'^6 V \\
&\quad + 3 (62208 + 22464 \Phi + 2196 \Phi^2 + 72 \Phi^3 + \Phi^4) \Phi'' V (\Phi'' + 8 V)^2 \\
&\quad + 2 (-684 - 48 \Phi + 11 \Phi^2) \Phi'^5 V' \\
&\quad + (6 + \Phi) \Phi' ((-31104 - 2772 \Phi + 120 \Phi^2 + 13 \Phi^3) \Phi''^2 \\
&\quad + 8 (-62208 - 7092 \Phi - 132 \Phi^2 + 7 \Phi^3) \Phi'' V \\
&\quad + 384 (-5184 - 504 \Phi + 6 \Phi^2 + \Phi^3) V^2) V' \\
&\quad - (6 + \Phi) \Phi'^3 (9 (-144 - 18 \Phi + \Phi^2) \Phi''' V \\
&\quad + (-3492 \Phi'' + 252 \Phi \Phi'' + 19 \Phi^2 \Phi'' \\
&\quad - 71712 V - 4944 \Phi V + 208 \Phi^2 V) V') \\
&\quad + 6 \Phi'^4 ((108 + 162 \Phi + 7 \Phi^2) \Phi'' V + 2 (-288 V^2 \\
&\quad + 504 \Phi V^2 + 36 \Phi^2 V^2 + 864 V'' + 252 \Phi V'' + 12 \Phi^2 V'' - \Phi^3 V'')) \\
&\quad - 6 \Phi'^2 ((6912 + 2736 \Phi + 192 \Phi^2 + \Phi^3) \Phi''^2 V \\
&\quad - 82944 V^3 + 14976 \Phi V^3 \\
&\quad + 3072 \Phi^2 V^3 + 160 \Phi^3 V^3 - 15552 \Phi''' V' \\
&\quad - 5400 \Phi \Phi''' V' - 468 \Phi^2 \Phi''' V' \\
&\quad + 6 \Phi^3 \Phi''' V' + \Phi^4 \Phi''' V' - 124416 V'^2 \\
&\quad - 43200 \Phi V'^2 - 3744 \Phi^2 V'^2 \\
&\quad + 48 \Phi^3 V'^2 + 8 \Phi^4 V'^2 + 124416 V V'' \\
&\quad + 43200 \Phi V V'' + 3744 \Phi^2 V V'' \\
&\quad - 48 \Phi^3 V V'' - 8 \Phi^4 V V'' \\
&\quad + \Phi'' (44928 V^2 + 24624 \Phi V^2 + 2208 \Phi^2 V^2 \\
&\quad + 52 \Phi^3 V^2 + 15552 V'' + 5400 \Phi V'')
\end{aligned}$$

$$\begin{aligned}
& +468 \Phi^2 V'' - 6 \Phi^3 V'' - \Phi^4 V'')))/ \\
& (8 (6 + \Phi)^2 (-2 \Phi'^2 + (24 + \Phi) \Phi'') (-2 \Phi'^2 \\
& + (18 + \Phi) (\Phi'' + 8 V))^2) \\
h_5 = & (\Phi' (-10 \Phi'^4 V + \Phi'^2 V ((426 + \Phi) \Phi'' - 8 (270 + \Phi) V) \\
& + \Phi \Phi'' V (-7 (6 + \Phi) \Phi'' + 8 (174 + 5 \Phi) V) + 12 (-24 + \Phi) \Phi'^3 V' \\
& + 6 (-432 - 6 \Phi + \Phi^2) \Phi' (\Phi''' V - \Phi'' V')))/ \\
& (4 (2 \Phi'^2 - (24 + \Phi) \Phi'') (-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V))^2) \\
h_6 = & (18 (-12 + 5 \Phi) \Phi'^6 V^2 + 4 (2772 + 384 \Phi - 13 \Phi^2) \Phi'^5 V V' \\
& - \Phi'' (3 (124416 + 44928 \Phi \\
& + 4212 \Phi^2 + 144 \Phi^3 + \Phi^4) \Phi''^2 V^2 \\
& + 48 \Phi'' (124416 V^3 + 44928 \Phi V^3 + 4212 \Phi^2 V^3 + 144 \Phi^3 V^3 + \Phi^4 V^3 \\
& - 23328 V'^2 - 10368 \Phi V'^2 - 1584 \Phi^2 V'^2 - 96 \Phi^3 V'^2 - 2 \Phi^4 V'^2) \\
& + 64 V (373248 V^3 + 134784 \Phi V^3 \\
& + 12636 \Phi^2 V^3 + 432 \Phi^3 V^3 + 3 \Phi^4 V^3 - 139968 V'^2 \\
& - 50544 \Phi V'^2 - 4320 \Phi^2 V'^2 + 216 \Phi^3 V'^2 + 36 \Phi^4 V'^2 + \Phi^5 V'^2)) \\
& - (6 + \Phi) \Phi'^3 (9 (-144 - 18 \Phi + \Phi^2) \Phi''' V^2 \\
& - 2 V' (-17244 \Phi'' V - 540 \Phi \Phi'' V + 29 \Phi^2 \Phi'' V - 99360 V^2 \\
& + 1992 \Phi V^2 + 212 \Phi^2 V^2 + 20736 V'' + 3744 \Phi V'' - 8 \Phi^3 V'')) \\
& + 2 (6 + \Phi) \Phi' V' ((62208 + 3708 \Phi - 24 \Phi^2 + \Phi^3) \Phi''^2 V \\
& - 4 \Phi'' (-248832 V^2 - 11736 \Phi V^2 + 840 \Phi^2 V^2 + 34 \Phi^3 V^2 \\
& + 46656 V'' + 11016 \Phi V'' + 468 \Phi^2 V'' - 18 \Phi^3 V'' - \Phi^4 V'')) \\
& - 2 (-432 - 6 \Phi + \Phi^2) (4608 V^3 + 192 \Phi V^3 \\
& + 108 \Phi''' V' + 24 \Phi \Phi''' V' + \Phi^2 \Phi''' V' + 864 V'^2 \\
& + 192 \Phi V'^2 + 8 \Phi^2 V'^2 - 1728 V V'' \\
& - 384 \Phi V V'' - 16 \Phi^2 V V'')) \\
& - 2 \Phi'^4 (3 (180 + 438 \Phi + 17 \Phi^2) \Phi'' V^2 + 4 (-4752 V^3 \\
& + 1116 \Phi V^3 + 66 \Phi^2 V^3 - 3240 V'^2 - 1008 \Phi V'^2 - 66 \Phi^2 V'^2 \\
& + 2 \Phi^3 V'^2 - 2592 V V'' - 756 \Phi V V'' - 36 \Phi^2 V V'' + 3 \Phi^3 V V'')) \\
& + 4 \Phi'^2 (6 (2484 + 1197 \Phi + 84 \Phi^2 + 2 \Phi^3) \Phi''^2 V^2 \\
& + \Phi'' (88128 V^3 + 67608 \Phi V^3 + 5040 \Phi^2 V^3 + 90 \Phi^3 V^3 - 125712 V'^2 \\
& - 46656 \Phi V'^2 - 4896 \Phi^2 V'^2 - 72 \Phi^3 V'^2 + 5 \Phi^4 V'^2 - 46656 V V'' \\
& - 16200 \Phi V V'' - 1404 \Phi^2 V V'' + 18 \Phi^3 V V'' + 3 \Phi^4 V V'')) \\
& + 3 V (-82944 V^3 + 30528 \Phi V^3 \\
& + 3840 \Phi^2 V^3 + 80 \Phi^3 V^3 + 15552 \Phi''' V' \\
& + 5400 \Phi \Phi''' V' + 468 \Phi^2 \Phi''' V' - 6 \Phi^3 \Phi''' V' - \Phi^4 \Phi''' V' + 72576 V'^2 \\
& + 28224 \Phi V'^2 + 3360 \Phi^2 V'^2 + 112 \Phi^3 V'^2 - 124416 V V'' \\
& - 43200 \Phi V V'' - 3744 \Phi^2 V V'' + 48 \Phi^3 V V'' + 8 \Phi^4 V V'')))/
\end{aligned}$$

$$\begin{aligned}
& (16 (6 + \Phi)^2 (-2 \Phi'^2 + (24 + \Phi) \Phi'') (-2 \Phi'^2 \\
& + (18 + \Phi) (\Phi'' + 8 V))^2) \\
h_7 = & - (V (84 \Phi^4 V - 8 (18 + \Phi)^2 \Phi'' V (-3 \Phi'' + 2 (-12 + \Phi) V) \\
& + \Phi'^2 V (3 (-876 - 40 \Phi + \Phi^2) \Phi'' + 8 (18 + \Phi)^2 V) \\
& - 4 (-432 - 6 \Phi + \Phi^2) \Phi'^3 V' \\
& - (-24 + \Phi) (18 + \Phi)^2 \Phi' (\Phi''' V - 2 (\Phi'' + 4 V) V')))) / \\
& ((2 \Phi'^2 - (24 + \Phi) \Phi'') (-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V))^2) \\
h_8 = & (-10 \Phi'^5 V^2 + 4 (-204 + 5 \Phi) \Phi'^4 V V' \\
& + 32 (18 + \Phi)^2 \Phi'' V (-3 \Phi'' + 2 (-12 + \Phi) V) V' \\
& + 2 \Phi'^2 V (3 (-432 - 6 \Phi + \Phi^2) \Phi''' V \\
& + (7416 \Phi'' + 270 \Phi \Phi'' - 11 \Phi^2 \Phi'' \\
& + 1728 V - 480 \Phi V - 32 \Phi^2 V) V') \\
& + \Phi'^3 ((426 + \Phi) \Phi'' V^2 - 8 (270 V^3 + \Phi V^3 \\
& + 432 V'^2 + 6 \Phi V'^2 - \Phi^2 V'^2 + 432 V V'' + 6 \Phi V V'' - \Phi^2 V V'')) \\
& + \Phi' (-6 \Phi (7 \Phi'^2 V^2 - 232 \Phi'' V^3 + 360 \Phi''' V V' \\
& - 360 \Phi'' V'^2 - 360 \Phi'' V V'' - 2880 V^2 V'') \\
& + 4 \Phi^3 (\Phi''' V V' - \Phi'' V'^2 - \Phi'' V V'' - 8 V^2 V'') \\
& + 31104 (-\Phi''' V V' + \Phi'' V'^2 + \Phi'' V V'' + 8 V^2 V'') \\
& - \Phi^2 (7 \Phi'^2 V^2 - 40 \Phi'' V^3 - 48 \Phi''' V V' + 48 \Phi'' V'^2 \\
& + 48 \Phi'' V V'' + 384 V^2 V'')))) / \\
& (4 (2 \Phi'^2 - (24 + \Phi) \Phi'') (-2 \Phi'^2 + (18 + \Phi) (\Phi'' + 8 V))^2) .
\end{aligned}$$

The c functions proposed in this thesis for  $d = 4$  case is given by  $h_1$  and  $h_2$  by putting  $\Phi'$  to vanish:

$$\begin{aligned}
c_1 &= \frac{2\pi}{3G} \frac{62208 + 22464\Phi + 2196\Phi^2 + 72\Phi^3 + \Phi^4}{(6 + \Phi)^2(24 + \Phi)(18 + \Phi)} \\
c_2 &= \frac{3\pi}{G} \frac{288 + 72\Phi + \Phi^2}{(6 + \Phi)^2(24 + \Phi)} \tag{A.6}
\end{aligned}$$

Note also that using of above condition on the zero value of dilatonic potential derivative on conformal boundary significantly simplifies the CA as many terms vanish.

## B Comparison with Other Counterterm Schemes

In this Appendix, we compare the counter terms and the trace anomaly obtained in section 4. with those in ref.[48, 49]. For simplicity, we consider the case that the spacetime dimension is 4 and the boundary is flat and the metric  $g_{ij}$  in (2.5) on the boundary is given by

$$g_{ij} = F(\rho)\eta_{ij} . \tag{B.1}$$

We also assume the dilaton  $\phi$  only depends on  $\rho$ :  $\phi = \phi(\rho)$ . This is exactly the case of ref.[48, 49]. Then the CA (2.22) vanishes on such background.

Let us demonstrate that this is consistent with results of ref.[48, 49]. In the metric (B.1), the equation of motion (2.2) given by the variation with respect to the dilaton  $\phi$  and the Einstein equations in (2.3) have the following forms:

$$0 = -\frac{l}{2\rho^3}F^2\Phi'(\phi) - \frac{2}{l}\partial_\rho\left(\frac{F^2}{\rho}\partial_\rho\phi\right) \quad (\text{B.2})$$

$$0 = \frac{l^2}{12\rho^2}\left(\Phi(\phi) + \frac{12}{l^2}\right) - \frac{1}{\rho^2} - \frac{2}{F}\partial_\rho^2 F + \frac{1}{F^2}(\partial_\rho F)^2 - \frac{1}{2}(\partial_\rho\phi)^2 \quad (\text{B.3})$$

$$0 = \frac{F}{3\rho}\left(\Phi(\phi) + \frac{12}{l^2}\right) - \frac{2\rho}{l^2}\partial_\rho^2 F - \frac{2\rho}{l^2 F}(\partial_\rho F)^2 + \frac{6}{l^2}\partial_\rho F - \frac{4F}{l^2\rho} . \quad (\text{B.4})$$

eq.(B.3) corresponds to  $\mu = \nu = \rho$  component in (2.3) and (B.4) to  $\mu = \nu = i$ . Other components equations in (2.3) vanish identically. Combining (B.3) and (B.4), we obtain

$$0 = -\frac{l^2}{4\rho^2}\left(\Phi(\phi) + \frac{12}{l^2}\right) + \frac{3}{\rho^2} + \frac{3}{F^2}(\partial_\rho F)^2 - \frac{6}{\rho F}\partial_\rho F - \frac{1}{2}(\partial_\rho\phi)^2 \quad (\text{B.5})$$

$$0 = -\frac{6}{F}\partial_\rho^2 F + \frac{6}{F^2}(\partial_\rho F)^2 - \frac{6}{\rho F}\partial_\rho F - 2(\partial_\rho\phi)^2 . \quad (\text{B.6})$$

If we define a new variable  $A$ , which corresponds to the exponent in the warp factor by

$$F = \rho e^{2A} , \quad (\text{B.7})$$

eq.(B.6) can be rewritten as

$$0 = -\frac{6}{\rho}\partial_\rho(\rho\partial_\rho A) - (\partial_\rho\phi)^2 . \quad (\text{B.8})$$

Now we further define a new variable  $B$  by

$$B \equiv \rho\partial_\rho A . \quad (\text{B.9})$$

If  $\frac{\partial\phi}{\partial\rho} \neq 0$ , we can regard  $B$  as a function of  $\phi$  instead of  $\rho$  and one obtains

$$\partial_\rho B = \frac{\partial B}{\partial\phi} \frac{\partial\phi}{\partial\rho} . \quad (\text{B.10})$$

By substituting (B.9) and (B.10) into (B.8), we find (by assuming  $\frac{\partial\phi}{\partial\rho} \neq 0$ )

$$\frac{\partial B}{\partial\phi} = -\frac{1}{6\rho}\partial_\rho\phi . \quad (\text{B.11})$$

Using (B.5) and (B.11) (and also (B.7) and (B.9)), we find that the dilaton equations motion (B.2) is automatically satisfied.

In [48, 49], another counterterms scheme is proposed

$$S_{\text{BGM}}^{(2)} = \frac{1}{16\pi G} \int d^4x \sqrt{-\hat{g}} \left\{ \frac{6u(\phi)}{l} + \frac{l}{2u(\phi)} R \right\} , \quad (\text{B.12})$$



instead of (4.7). Here  $u$  is obtained in terms of this thesis as follows:

$$u(\phi)^2 = 1 + \frac{l^2}{12}\Phi(\phi) . \quad (\text{B.13})$$

Then based on the counter terms in (B.12), the following expression of the trace anomaly is given in [48, 49]:<sup>7</sup>

$$T = \frac{3}{2\pi Gl}(-2B - u) . \quad (\text{B.14})$$

The above trace anomaly was evaluated for fixed but finite  $\rho$ . If the boundary is asymptotically AdS,  $F$  in (B.1) goes to a constant  $F \rightarrow F_0$  ( $F_0$ : a constant). Then from (B.7) and (B.9, we find the behaviors of  $A$  and  $B$  as

$$A \rightarrow \frac{1}{2} \ln \frac{F_0}{\rho} , \quad B \rightarrow -\frac{1}{2} . \quad (\text{B.15})$$

Then (B.11) tells that the dilaton  $\phi$  becomes a constant. Then (B.5) tells that

$$u = \sqrt{1 + \frac{l^2}{12}\Phi(\phi)} \rightarrow 1 . \quad (\text{B.16})$$

Eqs.(B.15) and (B.16) tell that the trace anomaly (B.14) vanishes on the boundary. Thus, we demonstrated that trace anomaly of [48, 49] vanishes in the UV limit what is expected also from AdS/CFT correspondence.

We should note that the trace anomaly (2.22) is evaluated on the boundary, i.e., in the UV limit. We evaluated the anomaly by expanding the action in the power series of  $\epsilon$  in (2.6) and subtracting the divergent terms in the limit of  $\epsilon \rightarrow 0$ . If we evaluate the anomaly for finite  $\rho$  as in [48, 49], the terms with positive power of  $\epsilon$  in the expansion do not vanish and we would obtain non-vanishing trace anomaly in general. Thus, the trace anomaly obtained in this thesis does not have any contradiction with that in [48, 49].

## C Remarks on Boundary Values

In section 2, from the leading order term in the equations of motion (2.2),

$$0 = -\sqrt{-\hat{G}} \frac{\partial \Phi(\phi_1, \dots, \phi_N)}{\partial \phi_\beta} - \partial_\mu \left( \sqrt{-\hat{G}} \hat{G}^{\mu\nu} \partial_\nu \phi_\beta \right) , \quad (\text{C.1})$$

which are given by variation of the action

$$S = \frac{1}{16\pi G} \int_{M_{d+1}} d^{d+1}x \sqrt{-\hat{G}} \left\{ \hat{R} - \sum_{\alpha=1}^N \frac{1}{2} (\hat{\nabla} \phi_\alpha)^2 + \Phi(\phi_1, \dots, \phi_N) + 4\lambda^2 \right\} . \quad (\text{C.2})$$

with respect to  $\phi_\alpha$ , we obtain

$$\frac{\partial \Phi(\phi_{(0)})}{\partial \phi_{(0)\alpha}} = 0. \quad (\text{C.3})$$

---

<sup>7</sup> The radial coordinate  $r$  in [48, 49] is related to  $\rho$  by  $dr = \frac{l d\rho}{4\rho}$ . Therefore  $\partial_r = -\frac{2\rho}{l} \partial_\rho$ , especially  $\partial_r A = -\frac{2\rho}{l} \partial_\rho A = -\frac{2}{l} B$ .

The equation (C.3) gives one of the necessary conditions that the spacetime is asymptotically AdS. The equation (C.3) also looks like a constraint that the boundary value  $\phi_{(0)}$  must take a special value satisfying (C.3) for the general fluctuations but it is not always correct. The condition  $\phi = \phi_{(0)}$  at the boundary is, of course, the boundary condition, which is not a part of the equations of motion. Due to the boundary condition, not all degrees of freedom of  $\phi$  are dynamical. Here the boundary value  $\phi_{(0)}$  is, of course, not dynamical. This tells that we should not impose the equations given only by the variation over  $\phi_{(0)}$ . The equation (C.3) is, in fact, only given by the variation of  $\phi_{(0)}$ . In order to understand the situation, we choose the metric in the following form

$$ds^2 \equiv \hat{G}_{\mu\nu} dx^\mu dx^\nu = \frac{l^2}{4} \rho^{-2} d\rho d\rho + \sum_{i=1}^d \hat{g}_{ij} dx^i dx^j, \quad \hat{g}_{ij} = \rho^{-1} g_{ij}, \quad (\text{C.4})$$

(If  $g_{ij} = \eta_{ij}$ , the boundary of AdS lies at  $\rho = 0$ .) and we use the regularization for the action (C.2) by introducing the infrared cutoff  $\epsilon$  and replacing

$$\int d^{d+1}x \rightarrow \int d^d x \int_\epsilon d\rho, \quad \int_{M_d} d^d x (\cdots) \rightarrow \int d^d x (\cdots) \Big|_{\rho=\epsilon}. \quad (\text{C.5})$$

Then the action (C.2) has the following form:

$$S = \frac{l}{16\pi G} \frac{1}{d} \epsilon^{-\frac{d}{2}} \int_{M_d} d^d x \sqrt{-\hat{g}_{(0)}} \left\{ \Phi(\phi_{1(0)}, \cdots, \phi_{N(0)}) - \frac{8}{l^2} \right\} + \mathcal{O}(\epsilon^{-\frac{d}{2}+1}). \quad (\text{C.6})$$

Then it is clear that eq.(C.3) can be derived only from the variation over  $\phi_{(0)}$  but not other components  $\phi_{(i)}$  ( $i = 1, 2, 3, \cdots$ ). Furthermore, if we add the surface counterterm  $S_b^{(1)}$

$$S_b^{(1)} = -\frac{1}{16\pi G} \frac{d}{2} \epsilon^{-\frac{d}{2}} \int_{M_d} d^d x \sqrt{-\hat{g}_{(0)}} \Phi(\phi_{1(0)}, \cdots, \phi_{N(0)}) \quad (\text{C.7})$$

to the action (C.2), the first  $\phi_{(0)}$  dependent term in (C.6) is cancelled and we find that eq.(C.3) cannot be derived from the variational principle. The surface counterterm in (C.7) is a part of the surface counterterms, which are necessary to obtain the well-defined AdS/CFT correspondence. Since the volume of AdS is infinite, the action (C.2) contains divergences, a part of which appears in (C.6). Then in order that we obtain the well-defined AdS/CFT set-up, we need the surface counterterms to cancel the divergence.

## D Scheme Dependence in AdS<sub>3</sub>/CFT<sub>2</sub>

In this Appendix, in order to show that scheme dependence takes place in other dimensions we consider calculation of 2-dimensional holographic CA from 3-dimensional dilatonic gravity with arbitrary bulk potential as the same way discussed in section 7. Using Hamilton-Jacobi formalism similarly to 5-dimensional gravity, we cast 3-dimensional ADM Hamiltonian density as (instead of (7.5))

$$\mathcal{H} \equiv \pi^2 - \pi_{\sigma\nu} \pi^{\sigma\nu} + \frac{\Pi^2}{2G} - \mathcal{L}. \quad (\text{D.1})$$

Hamiltonian constraint:  $\mathcal{H} = 0$ , leads to the following equation similar to (7.7)

$$\pi^2 - \pi_{\sigma\nu}\pi^{\sigma\nu} + \frac{\Pi^2}{2G} = \mathcal{R} + \frac{1}{2}G\gamma^{\sigma\nu}\partial_\sigma\phi\partial_\nu\phi + V \quad (\text{D.2})$$

We assume the form of 2-dimensional CA as

$$\langle T_\mu^\mu \rangle = \beta \langle O_\phi \rangle + cR, \quad (\text{D.3})$$

where  $R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$ . To solve the Hamilton-Jacobi equation (D.2), one uses the same procedure as in 4-dimensions. Substituting Hamilton momenta (7.9), (7.10) into (D.2), we obtain the relation between  $U$  and  $V$  from the potential term

$$\frac{U^2}{2} + \frac{U'^2}{2G} = V, \quad (\text{D.4})$$

and the curvature term  $R$  leads to the central charge  $c$

$$c = -\frac{2}{U} \left( 1 - \frac{Z'U'}{G} \right). \quad (\text{D.5})$$

We also obtain the following equation from  $R^2$  term:

$$\frac{c^2}{4} + \frac{Z'^2}{G} = 0. \quad (\text{D.6})$$

By deleting  $Z'$  from (D.5) by using (D.6), we find the following expression for the c-function  $c$

$$c = -\frac{2}{U \pm \frac{U'}{\sqrt{-G}}}. \quad (\text{D.7})$$

Especially if we choose constant potential  $V(\phi) = 2$  and  $G = -1$ , we find  $U$  and  $c$  from eqs.(D.4) and (D.7),

$$U = \pm 2, \quad c = \mp 1. \quad (\text{D.8})$$

Taking  $c = 1$ , this holographic RG result (at constant bulk potential) exactly agrees with the one of section 3.

Next we consider 3-dimensional bulk potential as

$$V = \frac{2}{\cosh^2 \phi}, \quad (\text{D.9})$$

with  $G = -1$ . Then by using (D.4) and (D.7), we obtain

$$U = -\frac{2}{\cosh \phi}, \quad c = e^{\pm \phi} \cosh^2 \phi. \quad (\text{D.10})$$

In section 2, we got c-function as

$$\begin{aligned} c_{\text{NOO}} &= \left( \frac{\mathcal{V}(\phi)}{2} + 2 \right) (\mathcal{V}(\phi) + 2)^{-1} \\ \mathcal{V}(\phi) &= V(\phi) - 2. \end{aligned} \quad (\text{D.11})$$

Substituting the potential (D.9) into (D.11), we obtained c-function as follows:

$$c_{\text{NOO}} = \frac{1 + \cosh^2 \phi}{2} . \quad (\text{D.12})$$

This c-function (D.12) does not coincide with (D.10) like in 4-dimensional case (apart from the leading, constant part). In (7.30) for 4-dimensional case and (D.11) for 2-dimensional case, the terms containing the derivatives of the potential  $V$  with respect to the scalar field  $\phi$  were neglected in section 2.3. As one might doubt that this might be the origin of the above disagreement, we investigate 2-dimensional case explicitly. If we include the neglected terms,  $c_{\text{NOO}}$  in (D.11) is modified as follows

$$\tilde{c}_{\text{NOO}} = 1 + \frac{\frac{2\mathcal{V}'(\phi)}{\mathcal{V}''(\phi)(\mathcal{V}(\phi)+2) - (\mathcal{V}'(\phi))^2} - \mathcal{V}(\phi)}{2(S\mathcal{V}(\phi) + 2)} . \quad (\text{D.13})$$

By substituting the potential (D.9) into (D.13), we obtained modified c-function  $\tilde{c}_{\text{NOO}}$  as follows:

$$\tilde{c}_{\text{NOO}} = \frac{1 + \cosh^2 \phi}{2} + \frac{1}{4} \sinh \phi \cosh^5 \phi . \quad (\text{D.14})$$

This c-function (D.14) does not coincide with (D.10) again.

## E The Calculations of Section 8

In this section we summarized the calculations of section 8. We show the detailed calculations of the bracket  $[\{S_1, S_2\}]$  for various weight.

First, we calculate  $[\{S_1, S_2\}]_{wt=6}$  from the combination  $[\mathcal{L}_{loc}]_2 = -\Phi R$  and  $[\mathcal{L}_{loc}]_4 = XR^2 + YR_{\mu\nu}R^{\mu\nu}$ . The  $X$  terms are given by

$$\begin{aligned} \frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=6} &= 2 \left( \frac{-1}{d-1} G_{\mu\nu} G_{\kappa\lambda} \frac{\delta S_1}{G_{\mu\nu}} \frac{\delta S_2}{G_{\kappa\lambda}} - G_{\mu\kappa} G_{\nu\lambda} \frac{\delta S_1}{G_{\mu\nu}} \frac{\delta S_2}{G_{\kappa\lambda}} \right) \\ &= -2X\Phi \left[ \frac{-1}{d-1} \left( \frac{d}{2} - 1 \right) R \left( \left( \frac{d}{2} - 2 \right) R^2 + 2(1-d)\nabla^2 R \right) \right. \\ &\quad \left. + \left\{ \frac{1}{2} R G_{\kappa\lambda} - R_{\kappa\lambda} \right\} \left\{ \frac{1}{2} R^2 G^{\kappa\lambda} + 2 \left( -RR^{\kappa\lambda} + \nabla^\kappa \nabla^\lambda R - G^{\kappa\lambda} \nabla^2 R \right) \right\} \right] \\ &= -2X\Phi \left[ \frac{-1}{d-1} \left( \left( \frac{d^2}{4} - \frac{3d}{2} + 2 \right) R^3 + (d-2)(1-d)R\nabla^2 R \right) \right. \\ &\quad \left. + \frac{d}{4} R^3 - R^3 + R\nabla^2 R - dR\nabla^2 R \right. \\ &\quad \left. - \frac{1}{2} R^3 + 2 \left( RR^{\kappa\lambda} R_{\kappa\lambda} - R_{\kappa\lambda} \nabla^\kappa \nabla^\lambda R + R\nabla^2 R \right) \right] \\ &= -2\Phi \left[ -\frac{d+2}{4(d-1)} R^3 + R\nabla^2 R - 2R_{\kappa\lambda} \nabla^\kappa \nabla^\lambda R + 2RR^{\kappa\lambda} R_{\kappa\lambda} \right] . \quad (\text{E.1}) \end{aligned}$$

The  $Y$  terms are given by

$$\begin{aligned}
&= -2Y\Phi\left[\frac{-1}{d-1}\left(\frac{d}{2}-1\right)R\left\{\left(\frac{d}{2}-2\right)R^{\mu\nu}R_{\mu\nu}-\frac{d}{2}\nabla^2R\right\}\right. \\
&\quad +\left\{\frac{1}{2}RG_{\kappa\lambda}-R_{\kappa\lambda}\right\}\left\{\frac{1}{2}R^{\mu\nu}R_{\mu\nu}G^{\kappa\lambda}+\nabla^\kappa\nabla^\lambda R\right. \\
&\quad \left.\left.-2G^{\kappa\omega}R_{\mu\omega\nu}^\lambda R^{\mu\nu}-\nabla^2R^{\kappa\lambda}-\frac{1}{2}G^{\kappa\lambda}\nabla^2R\right\}\right] \\
&= -2Y\Phi\left[\frac{-1}{d-1}\left\{\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-1\right)RR^{\mu\nu}R_{\mu\nu}-\frac{d}{2}\left(\frac{d}{2}-1\right)R\nabla^2R\right\}\right. \\
&\quad +\frac{d}{4}RR^{\mu\nu}R_{\mu\nu}+\frac{1}{2}R\nabla^2R-RR_{\mu\nu}R^{\mu\nu}-\frac{1}{2}R\nabla^2R-\frac{d}{4}R\nabla^2R \\
&\quad \left.-\frac{1}{2}RR^{\mu\nu}R_{\mu\nu}-R_{\kappa\lambda}\nabla^\kappa\nabla^\lambda R+2R_\lambda^\omega R_{\mu\omega\nu}^\lambda R^{\mu\nu}+R_{\kappa\lambda}\nabla^2R^{\kappa\lambda}+\frac{1}{2}R\nabla^2R\right] \\
&= -2Y\Phi\left[-\frac{d+2}{4(d-1)}RR^{\mu\nu}R_{\mu\nu}+\frac{d-2}{4(d-1)}R\nabla^2R\right. \\
&\quad \left.-R_{\kappa\lambda}\nabla^\kappa\nabla^\lambda R+2R_\lambda^\omega R_{\mu\omega\nu}^\lambda R^{\mu\nu}+R_{\kappa\lambda}\nabla^2R^{\kappa\lambda}\right]. \tag{E.2}
\end{aligned}$$

Then the contribution from the combination  $[\mathcal{L}_{loc}]_2 = -\Phi R$  and  $[\mathcal{L}_{loc}]_4 = XR^2 + YR_{\mu\nu}R^{\mu\nu}$  are as follows

$$\begin{aligned}
&[\{S_1, S_2\}]_{wt=6} \\
&= \Phi\left[\left(-4X+\frac{d+2}{2(d-1)}\right)RR_{\mu\nu}R^{\mu\nu}+\frac{d+2}{2(d-1)}XR^3-4YR^{\mu\lambda}R^{\nu\sigma}R_{\mu\nu\lambda\sigma}\right. \\
&\quad \left.+ (4X+2Y)R^{\mu\nu}\nabla_\mu\nabla_\nu R-2YR^{\mu\nu}\nabla^2R_{\mu\nu}+\left(-2X-\frac{d-2}{2(d-1)}Y\right)R\nabla^2R\right] \tag{E.3} \\
&\Phi = \frac{l}{d-2}, \quad X = \frac{dl^3}{4(d-1)(d-2)^2(d-4)}, \quad Y = -\frac{l^3}{(d-2)^2(d-4)}
\end{aligned}$$

Substituting above  $\Phi, X, Y$  (8.19) discussed in section 8 into (E.3), we get

$$\begin{aligned}
&[\{S_1, S_2\}]_{wt=6} \\
&= l^4\left[-\frac{3d+2}{2(d-1)(d-2)^3(d-4)}RR_{\mu\nu}R^{\mu\nu}+\frac{d(d+2)}{8(d-1)^2(d-2)^3(d-4)}R^3\right. \\
&\quad +\frac{4}{(d-2)^3(d-4)}R^{\mu\lambda}R^{\nu\sigma}R_{\mu\nu\lambda\sigma}-\frac{1}{(d-1)(d-2)^2(d-4)}R^{\mu\nu}\nabla_\mu\nabla_\nu R \\
&\quad \left.+\frac{2}{(d-2)^3(d-4)}R^{\mu\nu}\nabla^2R_{\mu\nu}-\frac{1}{(d-1)(d-2)^3(d-4)}R\nabla^2R\right] \tag{E.4}
\end{aligned}$$

This reproduce the result (8.20)

Next, we calculate  $[\{S_1, S_2\}]_{wt=6}$  from the combination  $[\mathcal{L}_{loc}]_1 = W$  and

$$\begin{aligned}
[\mathcal{L}_{loc}]_6 &= aR^3 + bRR_{\mu\nu}R^{\mu\nu} + cRR_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} + eR_{\mu\nu\lambda\sigma}R^{\mu\rho}R^{\nu\sigma} \\
&\quad + f\nabla_\mu R\nabla^\mu R + g\nabla_\mu R_{\nu\rho}\nabla^\mu R^{\nu\rho} + h\nabla_\mu R_{\nu\rho\sigma\tau}\nabla^\mu R^{\nu\rho\sigma\tau} + jR^{\mu\nu}R_\nu^\rho R_{\rho\mu}, \tag{E.5}
\end{aligned}$$

where  $W = -\frac{2(d-1)}{1}$ . From the combination  $\mathcal{L}_1 = W$  and  $\mathcal{L}_2 = aR^3$ , we get

$$\begin{aligned} \frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=6} &= \frac{-1}{d-1} G_{\mu\nu} G_{\kappa\lambda} \frac{\delta S_1}{G_{\mu\nu}} \frac{\delta S_2}{G_{\kappa\lambda}} - G_{\mu\kappa} G_{\nu\lambda} \frac{\delta S_1}{G_{\mu\nu}} \frac{\delta S_2}{G_{\kappa\lambda}} \\ &= aW \left[ \left( -\frac{d}{2} \frac{1}{d-1} + \frac{1}{2} \right) G_{\kappa\lambda} \frac{\delta S_2}{G_{\kappa\lambda}} \right] \\ &= \frac{-aW}{2(d-1)} \left\{ \left( \frac{d}{2} - 3 \right) R^3 + 6(1-d)(\nabla_\mu R \nabla^\mu R + R \nabla^2 R) \right\} \quad (\text{E.6}) \end{aligned}$$

From the combination  $\mathcal{L}_1 = W$  and  $\mathcal{L}_2 = bRR_{\mu\nu}R^{\mu\nu}$ , we get

$$\begin{aligned} \frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=6} &= \frac{-bW}{2(d-1)} \left\{ \left( \frac{d}{2} - 3 \right) RR_{\mu\nu}R^{\mu\nu} \right. \\ &\quad + 2(1-d)(\nabla_\alpha R_{\mu\nu} \nabla^\alpha R^{\mu\nu} + R^{\mu\nu} \nabla^2 R_{\mu\nu}) \\ &\quad \left. - \left( 1 + \frac{d}{2} \right) R \nabla^2 R + (2-d)R^{\mu\nu} \nabla_\mu \nabla_\nu R - d \nabla_\mu R \nabla^\mu R \right\}. \quad (\text{E.7}) \end{aligned}$$

From the combination  $\mathcal{L}_1 = W$  and  $\mathcal{L}_2 = cRR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ , we get

$$\begin{aligned} \frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=6} &= \frac{-cW}{2(d-1)} \left\{ \left( \frac{d}{2} - 3 \right) RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right. \\ &\quad + 2(1-d) \left( \nabla_\alpha R_{\mu\nu\rho\sigma} \nabla^\alpha R^{\mu\nu\rho\sigma} + R_{\mu\nu\rho\sigma} \nabla^2 R^{\mu\nu\rho\sigma} \right) \\ &\quad \left. - 4 \left( R_{\mu\nu} \nabla^\mu \nabla^\nu R + \nabla^\mu R \nabla_\mu R + \frac{1}{2} R \nabla^2 R \right) \right\}. \quad (\text{E.8}) \end{aligned}$$

From the combination  $\mathcal{L}_1 = W$  and  $\mathcal{L}_2 = eR_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma}$ , we get

$$\begin{aligned} \frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=6} &= \frac{-eW}{2(d-1)} \left\{ - \left( \frac{d}{2} + 2 \right) R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} - \frac{3}{4} \nabla_\mu R \nabla^\mu R \right. \\ &\quad + \left( \frac{d}{2} - 1 \right) R^{\mu\nu} \nabla_\mu \nabla_\nu R - \frac{1}{2} R \nabla^2 R + (2d-3) \nabla_\nu R^{\mu\rho} \nabla_\rho R^\nu_\mu \\ &\quad + (d-1) R^{\mu\rho} R^\nu_\mu R_{\nu\rho} + (2-d) R^{\mu\nu\rho\sigma} \nabla_\mu \nabla_\rho R^{\nu\sigma} \\ &\quad \left. - d R^{\mu\nu} \nabla^2 R_{\mu\nu} + 2(1-d) \nabla_\mu R^{\nu\rho} \nabla^\mu R_{\nu\rho} \right\}. \quad (\text{E.9}) \end{aligned}$$

From the combination  $\mathcal{L}_1 = W$  and  $\mathcal{L}_2 = f \nabla_\mu R \nabla^\mu R$ , we get

$$\frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=6} = \frac{-fW}{2(d-1)} \left\{ \left( \frac{d}{2} - 1 \right) \nabla_\mu R \nabla^\mu R + 2R \nabla^2 R + 2(d-1) \nabla^4 R \right\}. \quad (\text{E.10})$$

From the combination  $\mathcal{L}_1 = W$  and  $\mathcal{L}_2 = g \nabla_\mu R_{\nu\rho} \nabla^\mu R^{\nu\rho}$ , we get

$$\begin{aligned} \frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=6} &= \frac{-gW}{2(d-1)} \left\{ 2 \nabla_\nu R_{\mu\rho} \nabla^\mu R^{\nu\rho} + 2 R^{\mu\rho} R^\nu_\rho R_{\nu\mu} - 2 R^{\mu\rho} R^{\kappa\nu} R_{\nu\rho\kappa\mu} \right. \\ &\quad \left. + 2 R_{\nu\rho} \nabla^2 R^{\nu\rho} - \frac{1}{2} \nabla_\mu R \nabla^\mu R + \frac{d}{2} \nabla^4 R + \left( \frac{d}{2} - 1 \right) \nabla_\mu R_{\nu\rho} \nabla^\mu R^{\nu\rho} \right\}. \quad (\text{E.11}) \end{aligned}$$

From the combination  $\mathcal{L}_1 = W$  and  $\mathcal{L}_2 = h\nabla_\mu R_{\nu\rho\sigma\tau} \nabla^\mu R^{\nu\rho\sigma\tau}$ , we get

$$\begin{aligned} \frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=6} &= \frac{-hW}{2(d-1)} \left\{ \left( \frac{d}{2} - 1 \right) \nabla_\mu R_{\nu\rho\sigma\tau} \nabla^\mu R^{\nu\rho\sigma\tau} \right. \\ &\quad + 4\nabla_\nu R_{\mu\rho\sigma\tau} \nabla^\mu R^{\nu\rho\sigma\tau} + 2\nabla^4 R + 2R_{\nu\rho\sigma\tau} \nabla^2 R^{\nu\rho\sigma\tau} \\ &\quad + 8\nabla_\mu R_{\nu\rho} \nabla^\rho R^{\nu\mu} - 8\nabla_\mu R_{\nu\rho} \nabla^\mu R^{\nu\rho} \\ &\quad \left. + 4R^{\mu\rho\sigma\tau} R_{\lambda\mu} R_{\rho\sigma\tau}^\lambda - 4R^{\mu\rho\sigma\tau} R_{\mu\lambda\rho}^\nu R_{\nu\tau\sigma}^\lambda - 8R^{\mu\rho\sigma\tau} R_{\mu\lambda\sigma}^\nu R_{\tau\nu\rho}^\lambda \right\}. \end{aligned} \quad (\text{E.12})$$

From the combination  $\mathcal{L}_1 = W$  and  $\mathcal{L}_2 = jR^{\mu\rho} R_\rho^\nu R_{\nu\mu}$ , we get

$$\begin{aligned} \frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=6} &= \frac{-jW}{2(d-1)} \left\{ -dR^{\mu\rho} R_\rho^\nu R_{\nu\mu} + \frac{3}{2}(2-d) \left( R^{\mu\nu} \nabla_\mu \nabla_\nu R \right. \right. \\ &\quad \left. \left. + \frac{1}{4} \nabla^\mu R \nabla_\mu R + \nabla_\mu R^{\nu\rho} \nabla_\rho R_\nu^\mu - R^{\alpha\rho} R_\nu^\mu R_{\alpha\mu\rho}^\nu \right) \right. \\ &\quad \left. - 3 \left( R^{\mu\nu} \nabla^2 R_{\mu\nu} + \nabla^\mu R^{\nu\rho} \nabla_\mu R_{\nu\rho} \right) \right\}. \end{aligned} \quad (\text{E.13})$$

Adding the equation (E.4) to the summation of the equations from (E.6) to (E.13), we get all terms of  $[\{S_1, S_2\}]_{wt=6}$  as following form,

$$\begin{aligned} &[\{S_1, S_2\}]_{wt=6} \\ &= \left[ \left( b \left( \frac{d}{2} - 3 \right) \frac{2}{l} - \frac{(3d+2)l^4}{2(d-1)(d-2)^3(d-4)} \right) R R_{\mu\nu} R^{\mu\nu} \right. \\ &\quad + \left( a \left( \frac{d}{2} - 3 \right) \frac{2}{l} + \frac{d(d+2)l^4}{8(d-1)^2(d-2)^3(d-4)} \right) R^3 \\ &\quad + \left( \left\{ -e \left( \frac{d}{2} + 2 \right) - 2g - \frac{3j}{2}(2-d) \right\} \frac{2}{l} + \frac{4l^4}{(d-2)^3(d-4)} \right) R^{\mu\lambda} R^{\nu\sigma} R_{\mu\nu\lambda\sigma} \\ &\quad + \left( \left\{ b(2-d) - 4c + e \left( \frac{d}{2} - 1 \right) + j \frac{3}{2}(2-d) \right\} \frac{2}{l} - \frac{l^4}{(d-1)(d-2)^2(d-4)} \right) R^{\mu\nu} \nabla_\mu \nabla_\nu R \\ &\quad + \left( \{ 2b(1-d) - de + 2g - 3j \} \frac{2}{l} + \frac{2l^4}{(d-2)^3(d-4)} \right) R^{\mu\nu} \nabla^2 R_{\mu\nu} \\ &\quad + \left( \left\{ 6a(1-d) - b \left( 1 + \frac{d}{2} \right) - 2c - \frac{1}{2}e + 2f \right\} \frac{2}{l} - \frac{l^4}{(d-1)(d-2)^3(d-4)} \right) R \nabla^2 R \\ &\quad + \left( \frac{d}{2}g + 2h + 2f(d-1) \right) \frac{2}{l} \nabla^4 R + \left( \frac{d}{2} - 3 \right) \frac{2c}{l} R R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\ &\quad + \left( 6a(1-d) - db - 4c - \frac{3}{4}e + \left( \frac{d}{2} - 1 \right) f - \frac{g}{2} + \frac{3}{8}(2-d)j \right) \frac{2}{l} \nabla_\mu R \nabla^\mu R \\ &\quad + \left( 2b(1-d) + 2e(1-d) + g \left( \frac{d}{2} - 1 \right) - 8h - 3j \right) \frac{2}{l} \nabla_\kappa R_{\mu\nu} \nabla^\kappa R^{\mu\nu} \\ &\quad + \left( (2d-3)e + 2g + 8h + \frac{3}{2}(2-d)j \right) \frac{2}{l} \nabla_\kappa R^{\mu\nu} \nabla_\nu R_\mu^\kappa \end{aligned}$$

$$\begin{aligned}
& + ((d-1)e + 2g - dj) \frac{2}{l} R^{\mu\nu} R_\nu^\rho R_{\mu\rho} + (2-d) \frac{2e}{l} R^{\mu\nu\rho\sigma} \nabla_\mu \nabla_\rho R_{\nu\sigma} \\
& + \left( 2c(1-d) + \left( \frac{d}{2} - 1 \right) h \right) \frac{2}{l} \nabla_\alpha R_{\mu\nu\rho\sigma} \nabla^\alpha R^{\mu\nu\rho\sigma} + (2c(1-d) + 2h) \frac{2}{l} R_{\mu\nu\rho\sigma} \nabla^2 R^{\mu\nu\rho\sigma} \\
& + \left( 4R^{\mu\rho\sigma\tau} R_{\lambda\mu} R_{\rho\sigma\tau}^\lambda - 4R^{\mu\rho\sigma\tau} R_{\mu\lambda\rho}^\nu R_{\nu\tau\sigma}^\lambda \right) \frac{2h}{l} \\
& + \left( -8R^{\mu\rho\sigma\tau} R_{\mu\lambda\sigma}^\nu R_{\tau\nu\rho}^\lambda + 4\nabla_\nu R_{\mu\rho\sigma\tau} \nabla^\mu R^{\nu\rho\sigma\tau} \right) \frac{2h}{l}, \tag{E.14}
\end{aligned}$$

which reproduce the result (8.26).

Finally, we move on the calculations of  $[\{S_1, S_2\}]_{wt=8}$ . The contributions from the combination  $[\mathcal{L}_{loc}]_2 = -\Phi R$  where  $\Phi = \frac{l}{d-2}$ . and  $[\mathcal{L}_{loc}]_6$  as (E.5). From the combination  $\mathcal{L}_1 = -\Phi R$  and  $\mathcal{L}_2 = aR^3$ , we get

$$\begin{aligned}
\frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=8} &= G_{\mu\nu} G_{\kappa\lambda} \frac{\delta S_1}{G_{\mu\nu}} \frac{\delta S_2}{G_{\kappa\lambda}} - G_{\mu\kappa} G_{\nu\lambda} \frac{\delta S_1}{G_{\mu\nu}} \frac{\delta S_2}{G_{\kappa\lambda}} \\
&= -a\Phi \left\{ -\left( \frac{d}{2} - 1 \right) \frac{1}{d-1} + \frac{1}{2} \right\} R G_{\kappa\lambda} \frac{\delta S_2}{G_{\kappa\lambda}} + a\Phi R_{\kappa\lambda} \frac{\delta S_2}{G_{\kappa\lambda}} \\
&= -a\Phi \frac{1}{2(d-1)} R G_{\kappa\lambda} \frac{\delta S_2}{G_{\kappa\lambda}} + a\Phi R_{\kappa\lambda} \frac{\delta S_2}{G_{\kappa\lambda}} \\
&= -a\Phi \frac{1}{2(d-1)} R \left\{ \left( \frac{d}{2} - 3 \right) R^3 + 6(1-d)(\nabla_\mu R \nabla^\mu R + R \nabla^2 R) \right\} \\
&\quad + a\Phi R_{\kappa\lambda} \left\{ \frac{1}{2} G^{\kappa\lambda} R^3 + 3 \left( -R^{\kappa\lambda} R^2 + 2(\nabla^\kappa R \nabla^\lambda R + R \nabla^\kappa \nabla^\lambda R) \right. \right. \\
&\quad \left. \left. - 2G^{\kappa\lambda} (\nabla^\mu R \nabla_\mu R + R \nabla^2 R) \right) \right\} \\
&= -a\Phi \frac{1}{2(d-1)} R \left\{ \left( \frac{d}{2} - 3 \right) R^3 + 6(1-d)(\nabla_\mu R \nabla^\mu R + R \nabla^2 R) \right\} \\
&\quad + a\Phi \left\{ \frac{1}{2} R^4 + 3 \left( -R_{\kappa\lambda} R^{\kappa\lambda} R^2 + 2(R_{\kappa\lambda} \nabla^\kappa R \nabla^\lambda R + R_{\kappa\lambda} R \nabla^\kappa \nabla^\lambda R) \right. \right. \\
&\quad \left. \left. - 2R(\nabla^\mu R \nabla_\mu R + R \nabla^2 R) \right) \right\} \\
&= a\Phi \left\{ \frac{d+4}{4(d-1)} R^4 - 3R(\nabla^\mu R \nabla_\mu R + R \nabla^2 R) - 3R_{\kappa\lambda} R^{\kappa\lambda} R^2 \right. \\
&\quad \left. + 6(R_{\kappa\lambda} \nabla^\kappa R \nabla^\lambda R + R_{\kappa\lambda} R \nabla^\kappa \nabla^\lambda R) \right\}. \tag{E.15}
\end{aligned}$$

Substituting  $\Phi = \frac{l}{d-2}$ , the above equation is as follows;

$$\begin{aligned}
\frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=8} &= \frac{a}{2(d-2)} \left\{ \frac{d+4}{2(d-1)} R^4 - 6(R \nabla^\mu R \nabla_\mu R + R \nabla^2 R) \right. \\
&\quad \left. - 6R_{\kappa\lambda} R^{\kappa\lambda} R^2 + 12(R_{\kappa\lambda} \nabla^\kappa R \nabla^\lambda R + R_{\kappa\lambda} R \nabla^\kappa \nabla^\lambda R) \right\}. \tag{E.16}
\end{aligned}$$



From the combination  $\mathcal{L}_1 = -\Phi R$  and  $\mathcal{L}_2 = bRR_{\mu\nu}R^{\mu\nu}$ , we get

$$\begin{aligned}
\frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=8} &= -b\Phi \frac{1}{2(d-1)} RG_{\kappa\lambda} \frac{\delta S_2}{G_{\kappa\lambda}} + b\Phi R_{\kappa\lambda} \frac{\delta S_2}{G_{\kappa\lambda}} \\
&= -b \frac{\Phi R}{2(d-1)} \left\{ \left( \frac{d}{2} - 3 \right) RR_{\mu\nu}R^{\mu\nu} \right. \\
&\quad + 2(1-d)(\nabla_\alpha R_{\mu\nu} \nabla^\alpha R^{\mu\nu} + R^{\mu\nu} \nabla^2 R_{\mu\nu}) \\
&\quad - \left( 1 + \frac{d}{2} \right) R \nabla^2 R + (2-d) R^{\mu\nu} \nabla_\mu \nabla_\nu R - d \nabla_\mu R \nabla^\mu R \Big\} \\
&\quad + b\Phi R_{\kappa\lambda} \left\{ \frac{1}{2} G^{\kappa\lambda} RR_{\mu\nu}R^{\mu\nu} - R^{\kappa\lambda} R_{\mu\nu}R^{\mu\nu} \right. \\
&\quad - G^{\kappa\lambda} (2\nabla_\alpha R_{\mu\nu} \nabla^\alpha R^{\mu\nu} + 2R^{\mu\nu} \nabla^2 R_{\mu\nu}) \\
&\quad + 2R^{\mu\nu} \nabla^\kappa \nabla^\lambda R_{\mu\nu} + 2\nabla^\kappa R_{\mu\nu} \nabla^\lambda R^{\mu\nu} + 2R^{\mu\kappa} \nabla_\mu \nabla^\lambda R \\
&\quad + \nabla^\kappa R \nabla^\lambda R + 2\nabla_\mu R \nabla^\lambda R^{\kappa\mu} + R \nabla^\kappa \nabla^\lambda R - 2RR_{\alpha\mu} R^{\alpha\kappa\mu\lambda} \\
&\quad - R^{\kappa\lambda} \nabla^2 R - 2\nabla_\mu R \nabla^\mu R^{\kappa\lambda} - R \nabla^2 R^{\kappa\lambda} \\
&\quad \left. - G^{\kappa\lambda} \left( R^{\mu\nu} \nabla_\mu \nabla_\nu R + \nabla_\mu R \nabla^\mu R + \frac{1}{2} R \nabla^2 R \right) \right\} \\
&= -\frac{b\Phi}{2(d-1)} \left\{ \left( \frac{d}{2} - 3 \right) R^2 R_{\mu\nu}R^{\mu\nu} \right. \\
&\quad + 2(1-d)R(\nabla_\alpha R_{\mu\nu} \nabla^\alpha R^{\mu\nu} + R^{\mu\nu} \nabla^2 R_{\mu\nu}) \\
&\quad - \left( 1 + \frac{d}{2} \right) R^2 \nabla^2 R + (2-d)RR^{\mu\nu} \nabla_\mu \nabla_\nu R - dR \nabla_\mu R \nabla^\mu R \Big\} \\
&\quad + b\Phi \left\{ \frac{1}{2} R^2 R_{\mu\nu}R^{\mu\nu} - R_{\kappa\lambda} R^{\kappa\lambda} R_{\mu\nu}R^{\mu\nu} \right. \\
&\quad - R(2\nabla_\alpha R_{\mu\nu} \nabla^\alpha R^{\mu\nu} + 2R^{\mu\nu} \nabla^2 R_{\mu\nu}) \\
&\quad + 2R_{\kappa\lambda} R^{\mu\nu} \nabla^\kappa \nabla^\lambda R_{\mu\nu} + 2R_{\kappa\lambda} \nabla^\kappa R_{\mu\nu} \nabla^\lambda R^{\mu\nu} + 2R_{\kappa\lambda} R^{\mu\kappa} \nabla_\mu \nabla^\lambda R \\
&\quad + R_{\kappa\lambda} \nabla^\kappa R \nabla^\lambda R + 2R_{\kappa\lambda} \nabla_\mu R \nabla^\lambda R^{\kappa\mu} + R_{\kappa\lambda} R \nabla^\kappa \nabla^\lambda R \\
&\quad - 2RR_{\kappa\lambda} R_{\alpha\mu} R^{\alpha\kappa\mu\lambda} - R_{\kappa\lambda} R^{\kappa\lambda} \nabla^2 R - 2R_{\kappa\lambda} \nabla_\mu R \nabla^\mu R^{\kappa\lambda} \\
&\quad \left. - R_{\kappa\lambda} R \nabla^2 R^{\kappa\lambda} - R \left( R^{\mu\nu} \nabla_\mu \nabla_\nu R + \nabla_\mu R \nabla^\mu R + \frac{1}{2} R \nabla^2 R \right) \right\} \\
&= b\Phi \left\{ \frac{d+4}{4(d-1)} R^2 R_{\mu\nu}R^{\mu\nu} - R \nabla_\alpha R_{\mu\nu} \nabla^\alpha R^{\mu\nu} - 2RR^{\mu\nu} \nabla^2 R_{\mu\nu} \right. \\
&\quad + \frac{4-d}{4(d-1)} R^2 \nabla^2 R - \frac{2-d}{2(d-1)} RR^{\mu\nu} \nabla_\mu \nabla_\nu R - \frac{d-2}{2(d-1)} R \nabla_\mu R \nabla^\mu R \\
&\quad - \left( R_{\kappa\lambda} R^{\kappa\lambda} \right)^2 + 2R_{\kappa\lambda} R^{\mu\nu} \nabla^\kappa \nabla^\lambda R_{\mu\nu} + 2R_{\kappa\lambda} \nabla^\kappa R_{\mu\nu} \nabla^\lambda R^{\mu\nu} \\
&\quad + 2R_{\kappa\lambda} R^{\mu\kappa} \nabla_\mu \nabla^\lambda R + R_{\kappa\lambda} \nabla^\kappa R \nabla^\lambda R + 2R_{\kappa\lambda} \nabla_\mu R \nabla^\lambda R^{\kappa\mu} \\
&\quad \left. - 2RR_{\kappa\lambda} R_{\alpha\mu} R^{\alpha\kappa\mu\lambda} - R_{\kappa\lambda} R^{\kappa\lambda} \nabla^2 R - 2R_{\kappa\lambda} \nabla_\mu R \nabla^\mu R^{\kappa\lambda} \right\}. \quad (\text{E.17})
\end{aligned}$$

Substituting  $\Phi$ , the above equation is as follows;

$$\begin{aligned}
\frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=8} &= \frac{b}{2} \left\{ \frac{d+4}{2(d-1)(d-2)} R^2 R_{\mu\nu} R^{\mu\nu} - \frac{2}{d-2} R \nabla_\alpha R_{\mu\nu} \nabla^\alpha R^{\mu\nu} \right. \\
&\quad - \frac{4}{d-2} R R^{\mu\nu} \nabla^2 R_{\mu\nu} + \frac{4-d}{2(d-1)(d-2)} R^2 \nabla^2 R \\
&\quad + \frac{1}{(d-1)} R R^{\mu\nu} \nabla_\mu \nabla_\nu R - \frac{1}{(d-1)} R \nabla_\mu R \nabla^\mu R \\
&\quad + \frac{2}{d-2} \left[ - \left( R_{\kappa\lambda} R^{\kappa\lambda} \right)^2 + 2 R_{\kappa\lambda} R^{\mu\nu} \nabla^\kappa \nabla^\lambda R_{\mu\nu} + 2 R_{\kappa\lambda} \nabla^\kappa R_{\mu\nu} \nabla^\lambda R^{\mu\nu} \right. \\
&\quad + 2 R_{\kappa\lambda} R^{\mu\kappa} \nabla_\mu \nabla^\lambda R + R_{\kappa\lambda} \nabla^\kappa R \nabla^\lambda R + 2 R_{\kappa\lambda} \nabla_\mu R \nabla^\lambda R^{\kappa\mu} \\
&\quad \left. \left. - 2 R R_{\kappa\lambda} R_{\alpha\mu} R^{\alpha\kappa\mu\lambda} - R_{\kappa\lambda} R^{\kappa\lambda} \nabla^2 R - 2 R_{\kappa\lambda} \nabla_\mu R \nabla^\mu R^{\kappa\lambda} \right] \right\}. \quad (\text{E.18})
\end{aligned}$$

From the combination  $\mathcal{L}_1 = -\Phi R$  and  $\mathcal{L}_2 = e R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma}$ , we get

$$\begin{aligned}
\frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=8} &= -e\Phi \frac{1}{2(d-1)} R G_{\kappa\lambda} \frac{\delta S_2}{G_{\kappa\lambda}} + e\Phi R_{\kappa\lambda} \frac{\delta S_2}{G_{\kappa\lambda}} \\
&= e\Phi \left[ \frac{-R}{2(d-1)} \left\{ - \left( \frac{d}{2} + 2 \right) R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} - \frac{3}{4} \nabla_\mu R \nabla^\mu R \right. \right. \\
&\quad + \left( \frac{d}{2} - 1 \right) R^{\mu\nu} \nabla_\mu \nabla_\nu R - \frac{1}{2} R \nabla^2 R + (2d-3) \nabla_\nu R^{\mu\rho} \nabla_\rho R^\nu_\mu \\
&\quad + (d-1) R^{\mu\rho} R^\nu_\mu R_{\nu\rho} + (2-d) R^{\mu\nu\rho\sigma} \nabla_\mu \nabla_\rho R^{\nu\sigma} \\
&\quad \left. \left. - d R^{\mu\nu} \nabla^2 R_{\mu\nu} + 2(1-d) \nabla_\mu R^{\nu\rho} \nabla^\mu R_{\nu\rho} \right\} \right. \\
&\quad + R_{\kappa\lambda} \left\{ \frac{1}{2} G^{\kappa\lambda} R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} - 4 R^{\lambda\nu\rho\sigma} R^\kappa_\rho R_{\nu\sigma} \right. \\
&\quad + 2 R^\kappa_{\sigma\mu\nu} \nabla^\mu \nabla^\lambda R^{\nu\sigma} + 2 \nabla^\lambda R^{\mu\nu} \nabla^\kappa R_{\mu\nu} - 2 \nabla^\lambda R^{\mu\nu} \nabla_\mu R^\kappa_\nu \\
&\quad + 2 \nabla^\mu R^{\nu\sigma} \nabla^\lambda R^\kappa_{\sigma\mu\nu} + 2 R^{\nu\sigma} \nabla^\mu \nabla^\lambda R^\kappa_{\sigma\mu\nu} \\
&\quad - G^{\rho\lambda} \left( R^\kappa_{\nu\rho\sigma} \nabla^2 R^{\nu\sigma} + 2 \nabla^\mu R^{\nu\sigma} \nabla_\mu R^\kappa_{\nu\rho\sigma} + R^{\nu\sigma} \nabla^2 R^\kappa_{\nu\rho\sigma} \right) \\
&\quad - G^{\kappa\lambda} \left( R_{\mu\nu\rho\sigma} \nabla^\rho \nabla^\mu R^{\nu\sigma} + 2 \nabla^\mu R^{\nu\sigma} \nabla_\mu R_{\nu\sigma} - 2 \nabla^\mu R^{\nu\sigma} \nabla_\nu R_{\mu\sigma} \right. \\
&\quad \left. + R^{\mu\nu} \nabla^2 R_{\mu\nu} - \frac{1}{2} R^{\mu\nu} \nabla_\mu \nabla_\nu R - R^{\mu\nu} R^\rho_\mu R_{\nu\rho} + R^{\nu\sigma} R^\alpha_\rho R^\rho_{\sigma\alpha\nu} \right) \\
&\quad + \frac{1}{2} R^{\nu\kappa} \nabla_\nu \nabla^\lambda R + \frac{1}{4} \nabla_\kappa R \nabla^\lambda R + \nabla_\nu R^{\lambda\mu} \nabla_\mu R^{\nu\kappa} \\
&\quad + \frac{1}{2} R^{\lambda\mu} \nabla_\mu \nabla^\kappa R + R^{\lambda\rho} R^{\nu\alpha} R^\kappa_{\alpha\nu\rho} + R^{\lambda\rho} R^{\kappa\alpha} R_{\alpha\rho} \\
&\quad \left. \left. - \frac{1}{2} R^{\kappa\lambda} \nabla^2 R - \nabla^\mu R \nabla_\mu R^{\kappa\lambda} - R^{\mu\nu} \nabla_\mu \nabla_\nu R^{\kappa\lambda} + R^\lambda_{\nu\rho\sigma} R^{\kappa\rho} R^{\nu\sigma} \right\} \right] \\
&= e\Phi \left[ \frac{-R}{2(d-1)} \left\{ - \left( \frac{d}{2} + 2 \right) R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} - \frac{3}{4} \nabla_\mu R \nabla^\mu R \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \left( \frac{d}{2} - 1 \right) R^{\mu\nu} \nabla_\mu \nabla_\nu R - \frac{1}{2} R \nabla^2 R + (2d - 3) \nabla_\nu R^{\mu\rho} \nabla_\rho R^\nu_\mu \\
& + (d - 1) R^{\mu\rho} R^\nu_\mu R_{\nu\rho} + (2 - d) R^{\mu\nu\rho\sigma} \nabla_\mu \nabla_\rho R^{\nu\sigma} \\
& - d R^{\mu\nu} \nabla^2 R_{\mu\nu} + 2(1 - d) \nabla_\mu R^{\nu\rho} \nabla^\mu R_{\nu\rho} \Big\} \\
& + \frac{1}{2} R R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} - 4 R_{\kappa\lambda} R^{\lambda\nu\rho\sigma} R^\kappa_\rho R_{\nu\sigma} \\
& + 2 R_{\kappa\lambda} R^\kappa_{\sigma\mu\nu} \nabla^\mu \nabla^\lambda R^{\nu\sigma} + 2 R_{\kappa\lambda} \nabla^\lambda R^{\mu\nu} \nabla^\kappa R_{\mu\nu} - 2 R_{\kappa\lambda} \nabla^\lambda R^{\mu\nu} \nabla_\mu R^\kappa_\nu \\
& + 2 R_{\kappa\lambda} \nabla^\mu R^{\nu\sigma} \nabla^\lambda R^\kappa_{\sigma\mu\nu} + 2 R_{\kappa\lambda} R^{\nu\sigma} \nabla^\mu \nabla^\lambda R^\kappa_{\sigma\mu\nu} \\
& - R^\rho_\kappa \left( R^\kappa_{\nu\rho\sigma} \nabla^2 R^{\nu\sigma} + 2 \nabla^\mu R^{\nu\sigma} \nabla_\mu R^\kappa_{\nu\rho\sigma} + R^{\nu\sigma} \nabla^2 R^\kappa_{\nu\rho\sigma} \right) \\
& - R \left( R_{\mu\nu\rho\sigma} \nabla^\rho \nabla^\mu R^{\nu\sigma} + 2 \nabla^\mu R^{\nu\sigma} \nabla_\mu R_{\nu\sigma} - 2 \nabla^\mu R^{\nu\sigma} \nabla_\nu R_{\mu\sigma} \right. \\
& + R^{\mu\nu} \nabla^2 R_{\mu\nu} - \frac{1}{2} R^{\mu\nu} \nabla_\mu \nabla_\nu R - R^{\mu\nu} R^\rho_\mu R_{\nu\rho} + R^{\nu\sigma} R^\alpha_\rho R^\rho_{\sigma\alpha\nu} \Big) \\
& + \frac{1}{2} R_{\kappa\lambda} R^{\nu\kappa} \nabla_\nu \nabla^\lambda R + \frac{1}{4} R_{\kappa\lambda} \nabla_\kappa R \nabla^\lambda R + R_{\kappa\lambda} \nabla_\nu R^{\lambda\mu} \nabla_\mu R^{\nu\kappa} \\
& + \frac{1}{2} R_{\kappa\lambda} R^{\lambda\mu} \nabla_\mu \nabla^\kappa R + R_{\kappa\lambda} R^{\lambda\rho} R^{\nu\alpha} R^\kappa_{\alpha\nu\rho} + R_{\kappa\lambda} R^{\lambda\rho} R^{\kappa\alpha} R_{\alpha\rho} \\
& - \frac{1}{2} R_{\kappa\lambda} R^{\kappa\lambda} \nabla^2 R - R_{\kappa\lambda} \nabla^\mu R \nabla_\mu R^{\kappa\lambda} \\
& - R_{\kappa\lambda} R^{\mu\nu} \nabla_\mu \nabla_\nu R^{\kappa\lambda} + R_{\kappa\lambda} R^\lambda_{\nu\rho\sigma} R^{\kappa\rho} R^{\nu\sigma} \Big] \\
= & e\Phi \left[ \frac{-d+6}{4(d-1)} R R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} + \frac{3}{8(d-1)} R \nabla_\mu R \nabla^\mu R \right. \\
& + \frac{1}{4(d-1)} \left\{ d R R^{\mu\nu} \nabla_\mu \nabla_\nu R + R^2 \nabla^2 R + 2(2d-1) R \nabla_\nu R^{\mu\rho} \nabla_\rho R^\nu_\mu \right\} \\
& + \frac{1}{2} R R^{\mu\rho} R^\nu_\mu R_{\nu\rho} - \frac{d}{2(d-1)} R R^{\mu\nu\rho\sigma} \nabla_\mu \nabla_\rho R^{\nu\sigma} \\
& - \frac{d-2}{2(d-1)} R R^{\mu\nu} \nabla^2 R_{\mu\nu} - R \nabla_\mu R^{\nu\rho} \nabla^\mu R_{\nu\rho} \\
& - 4 R_{\kappa\lambda} R^{\lambda\nu\rho\sigma} R^\kappa_\rho R_{\nu\sigma} + 2 R_{\kappa\lambda} R^\kappa_{\sigma\mu\nu} \nabla^\mu \nabla^\lambda R^{\nu\sigma} \\
& + 2 R_{\kappa\lambda} \nabla^\lambda R^{\mu\nu} \nabla^\kappa R_{\mu\nu} - 2 R_{\kappa\lambda} \nabla^\lambda R^{\mu\nu} \nabla_\mu R^\kappa_\nu + 2 R_{\kappa\lambda} \nabla^\mu R^{\nu\sigma} \nabla^\lambda R^\kappa_{\sigma\mu\nu} \\
& + 2 \left( R_{\kappa\lambda} R^{\nu\sigma} \nabla^\lambda \nabla^\kappa R_{\nu\sigma} - R_{\kappa\lambda} R^{\nu\sigma} \nabla^\lambda \nabla_\sigma R^\kappa_\nu + R_{\kappa\lambda} R^{\nu\sigma} R^{\alpha\kappa\lambda\mu} R_{\alpha\sigma\mu\nu} \right. \\
& + R_{\kappa\lambda} R^\nu_\sigma R^{\alpha\sigma\lambda\mu} R^\kappa_{\alpha\mu\nu} + R_{\kappa\lambda} R^{\nu\sigma} R^{\alpha\lambda} R^\kappa_{\sigma\alpha\nu} + R_{\kappa\lambda} R^\sigma_\nu R^{\alpha\nu\lambda\mu} R^\kappa_{\sigma\mu\alpha} \Big) \\
& - R^\rho_\kappa \left( R^\kappa_{\nu\rho\sigma} \nabla^2 R^{\nu\sigma} + 2 \nabla^\mu R^{\nu\sigma} \nabla_\mu R^\kappa_{\nu\rho\sigma} + R^{\nu\sigma} \nabla^2 R^\kappa_{\nu\rho\sigma} \right) \\
& + \frac{1}{2} R_{\kappa\lambda} R^{\nu\kappa} \nabla_\nu \nabla^\lambda R + \frac{1}{4} R_{\kappa\lambda} \nabla_\kappa R \nabla^\lambda R + R_{\kappa\lambda} \nabla_\nu R^{\lambda\mu} \nabla_\mu R^{\nu\kappa} \\
& + \frac{1}{2} R_{\kappa\lambda} R^{\lambda\mu} \nabla_\mu \nabla^\kappa R + R_{\kappa\lambda} R^{\lambda\rho} R^{\kappa\alpha} R_{\alpha\rho} \Big]
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}R_{\kappa\lambda}R^{\kappa\lambda}\nabla^2 R - R_{\kappa\lambda}\nabla^\mu R\nabla_\mu R^{\kappa\lambda} - R_{\kappa\lambda}R^{\mu\nu}\nabla_\mu\nabla_\nu R^{\kappa\lambda} \Big] \\
= & e\Phi \left[ \frac{-d+6}{4(d-1)}RR_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} + \frac{3}{8(d-1)}R\nabla_\mu R\nabla^\mu R \right. \\
& + \frac{1}{4(d-1)} \left\{ dRR^{\mu\nu}\nabla_\mu\nabla_\nu R + R^2\nabla^2 R + 2(2d-1)R\nabla_\nu R^{\mu\rho}\nabla_\rho R^\nu_\mu \right\} \\
& + \frac{1}{2}RR^{\mu\rho}R^\nu_\mu R_{\nu\rho} - \frac{d}{2(d-1)}RR^{\mu\nu\rho\sigma}\nabla_\mu\nabla_\rho R^{\nu\sigma} \\
& - \frac{d-2}{2(d-1)}RR^{\mu\nu}\nabla^2 R_{\mu\nu} - R\nabla_\mu R^{\nu\rho}\nabla^\mu R_{\nu\rho} \\
& - 2R_{\kappa\lambda}R^{\lambda\nu\rho\sigma}R^\kappa_\rho R_{\nu\sigma} + 2R_{\kappa\lambda}R^\kappa_{\sigma\mu\nu}\nabla^\mu\nabla^\lambda R^{\nu\sigma} \\
& + 2R_{\kappa\lambda}\nabla^\lambda R^{\mu\nu}\nabla^\kappa R_{\mu\nu} - 2R_{\kappa\lambda}\nabla^\lambda R^{\mu\nu}\nabla_\mu R^\kappa_\nu + 2R_{\kappa\lambda}\nabla^\mu R^{\nu\sigma}\nabla^\lambda R^\kappa_{\sigma\mu\nu} \\
& + 2 \left( -R_{\kappa\lambda}R^{\nu\sigma}\nabla^\lambda\nabla_\sigma R^\kappa_\nu + R_{\kappa\lambda}R^{\nu\sigma}R^{\alpha\kappa\lambda\mu}R_{\alpha\sigma\mu\nu} \right. \\
& \left. + R_{\kappa\lambda}R^\nu_\sigma R^{\alpha\sigma\lambda\mu}R^\kappa_{\alpha\mu\nu} + R_{\kappa\lambda}R^\sigma_\nu R^{\alpha\nu\lambda\mu}R^\kappa_{\sigma\mu\alpha} \right) \\
& - R^\rho_\kappa \left( R^\kappa_{\nu\rho\sigma}\nabla^2 R^{\nu\sigma} + 2\nabla^\mu R^{\nu\sigma}\nabla_\mu R^\kappa_{\nu\rho\sigma} + R^{\nu\sigma}\nabla^2 R^\kappa_{\nu\rho\sigma} \right) \\
& + \frac{1}{4}R_{\kappa\lambda}\nabla_\kappa R\nabla^\lambda R + R_{\kappa\lambda}\nabla_\nu R^{\lambda\mu}\nabla_\mu R^{\nu\kappa} \\
& + R_{\kappa\lambda}R^{\lambda\mu}\nabla_\mu\nabla^\kappa R + R_{\kappa\lambda}R^{\lambda\rho}R^{\kappa\alpha}R_{\alpha\rho} \\
& \left. - \frac{1}{2}R_{\kappa\lambda}R^{\kappa\lambda}\nabla^2 R - R_{\kappa\lambda}\nabla^\mu R\nabla_\mu R^{\kappa\lambda} + R_{\kappa\lambda}R^{\mu\nu}\nabla_\mu\nabla_\nu R^{\kappa\lambda} \right]. \quad (\text{E.19})
\end{aligned}$$

Substituting  $\Phi$ , the above equation is as follows;

$$\begin{aligned}
\frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=8} = & \frac{e}{2} \left[ \frac{-d+6}{2(d-1)(d-2)}RR_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} + \frac{3}{4(d-1)(d-2)}R\nabla_\mu R\nabla^\mu R \right. \\
& + \frac{d}{2(d-1)(d-2)}RR^{\mu\nu}\nabla_\mu\nabla_\nu R + \frac{1}{2(d-1)(d-2)}R^2\nabla^2 R \\
& + \frac{2d-1}{(d-1)(d-2)}R\nabla_\nu R^{\mu\rho}\nabla_\rho R^\nu_\mu \\
& + \frac{1}{d-2}RR^{\mu\rho}R^\nu_\mu R_{\nu\rho} - \frac{d}{(d-1)(d-2)}RR^{\mu\nu\rho\sigma}\nabla_\mu\nabla_\rho R^{\nu\sigma} \\
& - \frac{1}{d-1}RR^{\mu\nu}\nabla^2 R_{\mu\nu} - \frac{2}{d-2}R\nabla_\mu R^{\nu\rho}\nabla^\mu R_{\nu\rho} \\
& + \frac{2}{d-2} \left\{ -2R_{\kappa\lambda}R^{\lambda\nu\rho\sigma}R^\kappa_\rho R_{\nu\sigma} + 2R_{\kappa\lambda}R^\kappa_{\sigma\mu\nu}\nabla^\mu\nabla^\lambda R^{\nu\sigma} \right. \\
& + 2R_{\kappa\lambda}\nabla^\lambda R^{\mu\nu}\nabla^\kappa R_{\mu\nu} - 2R_{\kappa\lambda}\nabla^\lambda R^{\mu\nu}\nabla_\mu R^\kappa_\nu + 2R_{\kappa\lambda}\nabla^\mu R^{\nu\sigma}\nabla^\lambda R^\kappa_{\sigma\mu\nu} \\
& + 2 \left( -R_{\kappa\lambda}R^{\nu\sigma}\nabla^\lambda\nabla_\sigma R^\kappa_\nu + R_{\kappa\lambda}R^{\nu\sigma}R^{\alpha\kappa\lambda\mu}R_{\alpha\sigma\mu\nu} \right. \\
& \left. + R_{\kappa\lambda}R^\nu_\sigma R^{\alpha\sigma\lambda\mu}R^\kappa_{\alpha\mu\nu} + R_{\kappa\lambda}R^\sigma_\nu R^{\alpha\nu\lambda\mu}R^\kappa_{\sigma\mu\alpha} \right) \\
& \left. - R^\rho_\kappa \left( R^\kappa_{\nu\rho\sigma}\nabla^2 R^{\nu\sigma} + 2\nabla^\mu R^{\nu\sigma}\nabla_\mu R^\kappa_{\nu\rho\sigma} + R^{\nu\sigma}\nabla^2 R^\kappa_{\nu\rho\sigma} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} R_{\kappa\lambda} \nabla_{\kappa} R \nabla^{\lambda} R + R_{\kappa\lambda} \nabla_{\nu} R^{\lambda\mu} \nabla_{\mu} R^{\nu\kappa} \\
& + R_{\kappa\lambda} R^{\lambda\mu} \nabla_{\mu} \nabla^{\kappa} R + R_{\kappa\lambda} R^{\lambda\rho} R^{\kappa\alpha} R_{\alpha\rho} \\
& - \frac{1}{2} R_{\kappa\lambda} R^{\kappa\lambda} \nabla^2 R - R_{\kappa\lambda} \nabla^{\mu} R \nabla_{\mu} R^{\kappa\lambda} + R_{\kappa\lambda} R^{\mu\nu} \nabla_{\mu} \nabla_{\nu} R^{\kappa\lambda} \Big\} \Big]. \quad (\text{E.20})
\end{aligned}$$

From the combination  $\mathcal{L}_1 = -\Phi R$  and  $\mathcal{L}_2 = f \nabla_{\mu} R \nabla^{\mu} R$ , we get

$$\begin{aligned}
\frac{1}{\sqrt{G}} [\{S_1, S_2\}]_{wt=8} &= -f\Phi \frac{1}{2(d-1)} R G_{\kappa\lambda} \frac{\delta S_2}{G_{\kappa\lambda}} + f\Phi R_{\kappa\lambda} \frac{\delta S_2}{G_{\kappa\lambda}} \\
&= f\Phi \left[ \frac{-R}{2(d-1)} \left\{ \left( \frac{d}{2} - 1 \right) \nabla_{\mu} R \nabla^{\mu} R + 2R \nabla^2 R + 2(d-1) \nabla^4 R \right\} \right. \\
&\quad + R_{\kappa\lambda} \left\{ \frac{1}{2} G^{\kappa\lambda} \nabla_{\mu} R \nabla^{\mu} R - \nabla_{\kappa} R \nabla^{\lambda} R \right. \\
&\quad \left. \left. + 2 \left( R^{\kappa\lambda} \nabla^2 R - \nabla^{\kappa} \nabla^{\lambda} \nabla^2 R + G^{\kappa\lambda} \nabla^4 R \right) \right\} \right] \\
&= f\Phi \left[ \frac{d}{4(d-1)} R \nabla_{\mu} R \nabla^{\mu} R - \frac{1}{d-1} R^2 \nabla^2 R + R \nabla^4 R \right. \\
&\quad \left. - R_{\kappa\lambda} \nabla_{\kappa} R \nabla^{\lambda} R + 2 \left( R_{\kappa\lambda} R^{\kappa\lambda} \nabla^2 R - R_{\kappa\lambda} \nabla^{\kappa} \nabla^{\lambda} \nabla^2 R \right) \right]. \quad (\text{E.21})
\end{aligned}$$

Substituting  $\Phi$ , the above equation is as follows;

$$\begin{aligned}
\frac{1}{\sqrt{G}} [\{S_1, S_2\}]_{wt=8} &= \frac{f}{2} \left[ \frac{d}{2(d-1)(d-2)} R \nabla_{\mu} R \nabla^{\mu} R - \frac{2}{(d-1)(d-2)} R^2 \nabla^2 R \right. \\
&\quad + \frac{2}{d-2} R \nabla^4 R - \frac{2}{d-2} R_{\kappa\lambda} \nabla_{\kappa} R \nabla^{\lambda} R \\
&\quad \left. + \frac{4}{d-2} \left( R_{\kappa\lambda} R^{\kappa\lambda} \nabla^2 R - R_{\kappa\lambda} \nabla^{\kappa} \nabla^{\lambda} \nabla^2 R \right) \right]. \quad (\text{E.22})
\end{aligned}$$

From the combination  $\mathcal{L}_1 = -\Phi R$  and  $\mathcal{L}_2 = g \nabla_{\mu} R_{\nu\rho} \nabla^{\mu} R^{\nu\rho}$ , we get

$$\begin{aligned}
\frac{1}{\sqrt{G}} [\{S_1, S_2\}]_{wt=8} &= -g\Phi \frac{1}{2(d-1)} R G_{\kappa\lambda} \frac{\delta S_2}{G_{\kappa\lambda}} + g\Phi R_{\kappa\lambda} \frac{\delta S_2}{G_{\kappa\lambda}} \\
&= g\Phi \left[ \frac{-R}{2(d-1)} \left\{ 2 \nabla_{\nu} R_{\mu\rho} \nabla^{\mu} R^{\nu\rho} + 2 R^{\mu\rho} R_{\rho}^{\nu} R_{\nu\mu} - 2 R^{\mu\rho} R^{\kappa\nu} R_{\nu\rho\kappa\mu} \right. \right. \\
&\quad + 2 R_{\nu\rho} \nabla^2 R^{\nu\rho} - \frac{1}{2} \nabla_{\mu} R \nabla^{\mu} R + \frac{d}{2} \nabla^4 R + \left( \frac{d}{2} - 1 \right) \nabla_{\mu} R_{\nu\rho} \nabla^{\mu} R^{\nu\rho} \Big\} \\
&\quad + R_{\kappa\lambda} \left\{ \frac{1}{2} G^{\kappa\lambda} \nabla_{\mu} R_{\nu\rho} \nabla^{\mu} R^{\nu\rho} - 2 \nabla_{\mu} R_{\rho}^{\kappa} \nabla^{\mu} R^{\lambda\rho} - \nabla^{\kappa} R_{\mu\nu} \nabla^{\lambda} R^{\mu\nu} \right. \\
&\quad - 2 \left( 2 \nabla^{\mu} R^{\kappa\nu} \nabla_{\mu} R_{\nu}^{\lambda} + R^{\kappa\nu} \nabla^2 R_{\nu}^{\lambda} + R_{\nu}^{\lambda} \nabla^2 R^{\kappa\nu} \right) \\
&\quad + 2 \left( 2 \nabla^{\mu} R_{\nu\alpha} \nabla_{\mu} R^{\alpha\kappa\nu\lambda} + R_{\nu\alpha} \nabla^2 R^{\alpha\kappa\nu\lambda} + R^{\alpha\kappa\nu\lambda} \nabla^2 R_{\nu\alpha} \right) \\
&\quad \left. \left. - \nabla^2 \nabla^{\kappa} \nabla^{\lambda} R + \nabla^4 R^{\kappa\lambda} + \frac{1}{2} G^{\kappa\lambda} \nabla^4 R \right\} \right]
\end{aligned}$$

$$\begin{aligned}
& +2\nabla_\nu R_\rho^\kappa \nabla^\lambda R^{\nu\rho} + R_\rho^\kappa \nabla^\lambda \nabla^\rho R + 2R_\rho^\kappa R_\mu^\rho R^{\mu\lambda} \\
& -2R_\rho^\kappa R_{\nu\mu} R^{\mu\rho\nu\lambda} + 2\nabla_\mu R_\rho^\kappa \nabla^\mu R^{\lambda\rho} + 2R_\mu^\kappa \nabla^2 R^{\lambda\mu} \\
& -\nabla_\mu R \nabla^\lambda R^{\kappa\mu} - 2R_\nu^\mu \nabla_\mu \nabla^\lambda R^{\kappa\nu} \Big\} \Big] \\
= & g\Phi \Big[ \frac{-R}{2(d-1)} \Big\{ 2\nabla_\nu R_{\mu\rho} \nabla^\mu R^{\nu\rho} + 2R^{\mu\rho} R_\rho^\nu R_{\nu\mu} - 2R^{\mu\rho} R^{\kappa\nu} R_{\nu\rho\kappa\mu} \\
& + 2R_{\nu\rho} \nabla^2 R^{\nu\rho} - \frac{1}{2} \nabla_\mu R \nabla^\mu R + \frac{d}{2} \nabla^4 R + \left( \frac{d}{2} - 1 \right) \nabla_\mu R_{\nu\rho} \nabla^\mu R^{\nu\rho} \Big\} \\
& + \left\{ \frac{1}{2} R \nabla_\mu R_{\nu\rho} \nabla^\mu R^{\nu\rho} - 2R_{\kappa\lambda} \nabla_\mu R_\rho^\kappa \nabla^\mu R^{\lambda\rho} - R_{\kappa\lambda} \nabla^\kappa R_{\mu\nu} \nabla^\lambda R^{\mu\nu} \right. \\
& - 4R_{\kappa\lambda} R^{\kappa\nu} \nabla^2 R_\nu^\lambda + 2 \left( 2R_{\kappa\lambda} \nabla^\mu R_{\nu\alpha} \nabla_\mu R^{\alpha\kappa\nu\lambda} \right. \\
& \left. + R_{\kappa\lambda} R_{\nu\alpha} \nabla^2 R^{\alpha\kappa\nu\lambda} + R_{\kappa\lambda} R^{\alpha\kappa\nu\lambda} \nabla^2 R_{\nu\alpha} \right) \\
& - R_{\kappa\lambda} \nabla^2 \nabla^\kappa \nabla^\lambda R + R_{\kappa\lambda} \nabla^4 R^{\kappa\lambda} + \frac{1}{2} R \nabla^4 R \\
& + 2R_{\kappa\lambda} \nabla_\nu R_\rho^\kappa \nabla^\lambda R^{\nu\rho} + R_{\kappa\lambda} R_\rho^\kappa \nabla^\lambda \nabla^\rho R + 2R_{\kappa\lambda} R_\rho^\kappa R_\mu^\rho R^{\mu\lambda} \\
& - 2R_{\kappa\lambda} R_\rho^\kappa R_{\nu\mu} R^{\mu\rho\nu\lambda} - 2R_{\kappa\lambda} \nabla_\mu R_\rho^\kappa \nabla^\mu R^{\lambda\rho} + 2R_{\kappa\lambda} R_\mu^\kappa \nabla^2 R^{\lambda\mu} \\
& \left. - R_{\kappa\lambda} \nabla_\mu R \nabla^\lambda R^{\kappa\mu} - 2R_{\kappa\lambda} R_\nu^\mu \nabla_\mu \nabla^\lambda R^{\kappa\nu} \right\} \Big] \\
= & g\Phi \Big[ -\frac{1}{d-1} \Big\{ R \nabla_\nu R_{\mu\rho} \nabla^\mu R^{\nu\rho} + R R^{\mu\rho} R_\rho^\nu R_{\nu\mu} - R R^{\mu\rho} R^{\kappa\nu} R_{\nu\rho\kappa\mu} \Big\} \\
& - \frac{1}{d-1} R R_{\nu\rho} \nabla^2 R^{\nu\rho} + \frac{1}{4(d-1)} R \nabla_\mu R \nabla^\mu R + \frac{d-2}{4(d-1)} R \nabla^4 R \\
& + \frac{d}{4(d-1)} R \nabla_\mu R_{\nu\rho} \nabla^\mu R^{\nu\rho} + \left\{ -4R_{\kappa\lambda} \nabla_\mu R_\rho^\kappa \nabla^\mu R^{\lambda\rho} \right. \\
& - R_{\kappa\lambda} \nabla^\kappa R_{\mu\nu} \nabla^\lambda R^{\mu\nu} - 4R_{\kappa\lambda} R^{\kappa\nu} \nabla^2 R_\nu^\lambda + 2 \left( 2R_{\kappa\lambda} \nabla^\mu R_{\nu\alpha} \nabla_\mu R^{\alpha\kappa\nu\lambda} \right. \\
& \left. + R_{\kappa\lambda} R_{\nu\alpha} \nabla^2 R^{\alpha\kappa\nu\lambda} + R_{\kappa\lambda} R^{\alpha\kappa\nu\lambda} \nabla^2 R_{\nu\alpha} \right) \\
& - R_{\kappa\lambda} \nabla^2 \nabla^\kappa \nabla^\lambda R + R_{\kappa\lambda} \nabla^4 R^{\kappa\lambda} \\
& + 2R_{\kappa\lambda} \nabla_\nu R_\rho^\kappa \nabla^\lambda R^{\nu\rho} + R_{\kappa\lambda} R_\rho^\kappa \nabla^\lambda \nabla^\rho R + 2R_{\kappa\lambda} R_\rho^\kappa R_\mu^\rho R^{\mu\lambda} \\
& - 2R_{\kappa\lambda} R_\rho^\kappa R_{\nu\mu} R^{\mu\rho\nu\lambda} + 2R_{\kappa\lambda} R_\mu^\kappa \nabla^2 R^{\lambda\mu} \\
& \left. - R_{\kappa\lambda} \nabla_\mu R \nabla^\lambda R^{\kappa\mu} - 2R_{\kappa\lambda} R_\nu^\mu \nabla_\mu \nabla^\lambda R^{\kappa\nu} \right\} \Big]. \tag{E.23}
\end{aligned}$$

Substituting  $\Phi$ , the above equation is as follows;

$$\begin{aligned}
\frac{1}{\sqrt{G}} [\{S_1, S_2\}]_{wt=8} &= \frac{g}{2} \Big[ -\frac{2}{(d-1)(d-2)} R \nabla_\nu R_{\mu\rho} \nabla^\mu R^{\nu\rho} - \frac{2}{(d-1)(d-2)} R R^{\mu\rho} R_\rho^\nu R_{\nu\mu} \\
& + \frac{2}{(d-1)(d-2)} R R^{\mu\rho} R^{\kappa\nu} R_{\nu\rho\kappa\mu} - \frac{2}{(d-1)(d-2)} R R_{\nu\rho} \nabla^2 R^{\nu\rho} \\
& + \frac{1}{2(d-1)(d-2)} R \nabla_\mu R \nabla^\mu R + \frac{1}{2(d-1)} R \nabla^4 R
\end{aligned}$$

$$\begin{aligned}
& + \frac{d}{2(d-1)(d-2)} R \nabla_\mu R_{\nu\rho} \nabla^\mu R^{\nu\rho} \\
& + \frac{2}{d-2} \left\{ -4R_{\kappa\lambda} \nabla_\mu R_\rho^\kappa \nabla^\mu R^{\lambda\rho} - R_{\kappa\lambda} \nabla^\kappa R_{\mu\nu} \nabla^\lambda R^{\mu\nu} \right. \\
& - 4R_{\kappa\lambda} R^{\kappa\nu} \nabla^2 R_\nu^\lambda + 2 \left( 2R_{\kappa\lambda} \nabla^\mu R_{\nu\alpha} \nabla_\mu R^{\alpha\kappa\nu\lambda} \right. \\
& + R_{\kappa\lambda} R_{\nu\alpha} \nabla^2 R^{\alpha\kappa\nu\lambda} + R_{\kappa\lambda} R^{\alpha\kappa\nu\lambda} \nabla^2 R_{\nu\alpha} \Big) \\
& - R_{\kappa\lambda} \nabla^2 \nabla^\kappa \nabla^\lambda R + R_{\kappa\lambda} \nabla^4 R^{\kappa\lambda} \\
& + 2R_{\kappa\lambda} \nabla_\nu R_\rho^\kappa \nabla^\lambda R^{\nu\rho} + R_{\kappa\lambda} R_\rho^\kappa \nabla^\lambda \nabla^\rho R + 2R_{\kappa\lambda} R_\rho^\kappa R_\mu^\rho R^{\mu\lambda} \\
& - 2R_{\kappa\lambda} R_\rho^\kappa R_{\nu\mu} R^{\mu\rho\nu\lambda} + 2R_{\kappa\lambda} R_\mu^\kappa \nabla^2 R^{\lambda\mu} \\
& \left. - R_{\kappa\lambda} \nabla_\mu R \nabla^\lambda R^{\kappa\mu} - 2R_{\kappa\lambda} R_\nu^\mu \nabla_\mu \nabla^\lambda R^{\kappa\nu} \right\} \Big]. \tag{E.24}
\end{aligned}$$

Thus the contribution from  $\mathcal{L}_1 = -\Phi R$  and

$$\begin{aligned}
\mathcal{L}_6 = & aR^3 + bRR_{\mu\nu}R^{\mu\nu} + cRR_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} + eR_{\mu\nu\lambda\sigma}R^{\mu\rho}R^{\nu\sigma} \\
& + f\nabla_\mu R \nabla^\mu R + g\nabla_\mu R_{\nu\rho} \nabla^\mu R^{\nu\rho} + h\nabla_\mu R_{\nu\rho\sigma\tau} \nabla^\mu R^{\nu\rho\sigma\tau} + jR^{\mu\nu} R_\nu^\rho R_{\rho\mu}
\end{aligned} \tag{E.25}$$

for the calculations of  $[\{S_1, S_2\}]_{wt=8}$  are following form.

$$\begin{aligned}
\frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=8} = & a \left[ \frac{d+4}{2(d-1)(d-2)} R^4 - \frac{6}{d-2} R (\nabla^\mu R \nabla_\mu R + R \nabla^2 R) \right. \\
& - \frac{6}{d-2} R_{\kappa\lambda} R^{\kappa\lambda} R^2 + \frac{12}{d-2} (R_{\kappa\lambda} \nabla^\kappa R \nabla^\lambda R + R_{\kappa\lambda} R \nabla^\kappa \nabla^\lambda R) \Big] \\
& + b \left[ \frac{1}{d-2} \left\{ \frac{d+4}{2(d-1)} R^2 R_{\mu\nu} R^{\mu\nu} - 2R \nabla_\alpha R_{\mu\nu} \nabla^\alpha R^{\mu\nu} \right. \right. \\
& - 4RR^{\mu\nu} \nabla^2 R_{\mu\nu} + \frac{4-d}{2(d-1)} R^2 \nabla^2 R \\
& + \frac{d-2}{d-1} \{ RR^{\mu\nu} \nabla_\mu \nabla_\nu R - R \nabla_\mu R \nabla^\mu R \} \\
& + 2 \left( - (R_{\kappa\lambda} R^{\kappa\lambda})^2 + 2R_{\kappa\lambda} R^{\mu\nu} \nabla^\kappa \nabla^\lambda R_{\mu\nu} + 2R_{\kappa\lambda} \nabla^\kappa R_{\mu\nu} \nabla^\lambda R^{\mu\nu} \right. \\
& + 2R_{\kappa\lambda} R^{\mu\kappa} \nabla_\mu \nabla^\lambda R + R_{\kappa\lambda} \nabla^\kappa R \nabla^\lambda R + 2R_{\kappa\lambda} \nabla_\mu R \nabla^\lambda R^{\kappa\mu} \\
& \left. \left. - 2RR_{\kappa\lambda} R_{\alpha\mu} R^{\alpha\kappa\mu\lambda} - R_{\kappa\lambda} R^{\kappa\lambda} \nabla^2 R - 2R_{\kappa\lambda} \nabla_\mu R \nabla^\mu R^{\kappa\lambda} \right) \right\} \Big] \\
& + e \left[ \frac{1}{2(d-1)(d-2)} \left\{ (-d+6) RR_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} + \frac{3}{2} R \nabla_\mu R \nabla^\mu R \right. \right. \\
& + dRR^{\mu\nu} \nabla_\mu \nabla_\nu R + R^2 \nabla^2 R + 2(2d-1) R \nabla_\nu R^{\mu\rho} \nabla_\rho R_\mu^\nu \\
& + 2(d-1) RR^{\mu\rho} R_\mu^\nu R_{\nu\rho} - 2dRR^{\mu\nu\rho\sigma} \nabla_\mu \nabla_\rho R^{\nu\sigma} \\
& - 2(d-2) RR^{\mu\nu} \nabla^2 R_{\mu\nu} - 4(d-1) R \nabla_\mu R^{\nu\rho} \nabla^\mu R_{\nu\rho} \\
& \left. + 4(d-1) \left( -2R_{\kappa\lambda} R^{\lambda\nu\rho\sigma} R_\rho^\kappa R_{\nu\sigma} + 2R_{\kappa\lambda} R_{\sigma\mu\nu}^\kappa \nabla^\mu \nabla^\lambda R^{\nu\sigma} \right) \right\} \Big]
\end{aligned}$$

$$\begin{aligned}
& +2R_{\kappa\lambda}\nabla^\lambda R^{\mu\nu}\nabla^\kappa R_{\mu\nu} - 2R_{\kappa\lambda}\nabla^\lambda R^{\mu\nu}\nabla_\mu R_\nu^\kappa + 2R_{\kappa\lambda}\nabla^\mu R^{\nu\sigma}\nabla^\lambda R_{\sigma\mu}^\kappa \\
& +2\left(-R_{\kappa\lambda}R^{\nu\sigma}\nabla^\lambda\nabla_\sigma R_\nu^\kappa + R_{\kappa\lambda}R^{\nu\sigma}R^{\alpha\kappa\lambda\mu}R_{\alpha\sigma\mu\nu}\right. \\
& +R_{\kappa\lambda}R_\sigma^\nu R^{\alpha\sigma\lambda\mu}R_{\alpha\mu\nu}^\kappa + R_{\kappa\lambda}R_\nu^\sigma R^{\alpha\nu\lambda\mu}R_{\sigma\mu\alpha}^\kappa) \\
& -R_\kappa^\rho\left(R_{\nu\rho\sigma}^\kappa\nabla^2 R^{\nu\sigma} + 2\nabla^\mu R^{\nu\sigma}\nabla_\mu R_{\nu\rho\sigma}^\kappa + R^{\nu\sigma}\nabla^2 R_{\nu\rho\sigma}^\kappa\right) \\
& +\frac{1}{4}R_{\kappa\lambda}\nabla_\kappa R\nabla^\lambda R + R_{\kappa\lambda}\nabla_\nu R^{\lambda\mu}\nabla_\mu R^{\nu\kappa} \\
& +R_{\kappa\lambda}R^{\lambda\mu}\nabla_\mu\nabla^\kappa R + R_{\kappa\lambda}R^{\lambda\rho}R^{\kappa\alpha}R_{\alpha\rho} \\
& \left.-\frac{1}{2}R_{\kappa\lambda}R^{\kappa\lambda}\nabla^2 R - R_{\kappa\lambda}\nabla^\mu R\nabla_\mu R^{\kappa\lambda} + R_{\kappa\lambda}R^{\mu\nu}\nabla_\mu\nabla_\nu R^{\kappa\lambda}\right)\Bigg] \\
& +f\left[\frac{1}{(d-1)(d-2)}\left\{\frac{d}{2}R\nabla_\mu R\nabla^\mu R - 2R^2\nabla^2 R\right.\right. \\
& +2(d-1)R\nabla^4 R - 2(d-1)R_{\kappa\lambda}\nabla_\kappa R\nabla^\lambda R \\
& \left.+4(d-1)\left(R_{\kappa\lambda}R^{\kappa\lambda}\nabla^2 R - R_{\kappa\lambda}\nabla^\kappa\nabla^\lambda\nabla^2 R\right)\right\}\Bigg] \\
& +g\left[\frac{1}{(d-1)(d-2)}\left\{-2R\nabla_\nu R_{\mu\rho}\nabla^\mu R^{\nu\rho} - 2RR^{\mu\rho}R_\rho^\nu R_{\nu\mu}\right.\right. \\
& +2RR^{\mu\rho}R^{\kappa\nu}R_{\nu\rho\kappa\mu} - 2RR_{\nu\rho}\nabla^2 R^{\nu\rho} \\
& +\frac{1}{2}R\nabla_\mu R\nabla^\mu R + \frac{d-2}{2}R\nabla^4 R + \frac{d}{2}R\nabla_\mu R_{\nu\rho}\nabla^\mu R^{\nu\rho} \\
& +2(d-1)\left(-4R_{\kappa\lambda}\nabla_\mu R_\rho^\kappa\nabla^\mu R^{\lambda\rho} - R_{\kappa\lambda}\nabla^\kappa R_{\mu\nu}\nabla^\lambda R^{\mu\nu}\right. \\
& -4R_{\kappa\lambda}R^{\kappa\nu}\nabla^2 R_\nu^\lambda + 2\left(2R_{\kappa\lambda}\nabla^\mu R_{\nu\alpha}\nabla_\mu R^{\alpha\kappa\nu\lambda}\right. \\
& +R_{\kappa\lambda}R_{\nu\alpha}\nabla^2 R^{\alpha\kappa\nu\lambda} + R_{\kappa\lambda}R^{\alpha\kappa\nu\lambda}\nabla^2 R_{\nu\alpha}) \\
& -R_{\kappa\lambda}\nabla^2\nabla^\kappa\nabla^\lambda R + R_{\kappa\lambda}\nabla^4 R^{\kappa\lambda} \\
& +2R_{\kappa\lambda}\nabla_\nu R_\rho^\kappa\nabla^\lambda R^{\nu\rho} + R_{\kappa\lambda}R_\rho^\kappa\nabla^\lambda\nabla^\rho R + 2R_{\kappa\lambda}R_\rho^\kappa R_\mu^\rho R^{\mu\lambda} \\
& -2R_{\kappa\lambda}R_\rho^\kappa R_{\nu\mu}R^{\mu\rho\nu\lambda} + 2R_{\kappa\lambda}R_\mu^\kappa\nabla^2 R^{\lambda\mu} \\
& \left.-R_{\kappa\lambda}\nabla_\mu R\nabla^\lambda R^{\kappa\mu} - 2R_{\kappa\lambda}R_\nu^\mu\nabla_\mu\nabla^\lambda R^{\kappa\nu}\right)\Bigg] \\
& = \frac{1}{(d-1)(d-2)}\left\{\frac{a(d+4)}{2}R^4\right. \\
& +\left(-6a(d-1) + \frac{(4-d)b}{2} + \frac{e}{2} - 2f\right)R^2\nabla^2 R \\
& +\left(-6a(d-1) - b(d-2) + \frac{3e}{4} + \frac{df}{2} + \frac{g}{2}\right)R\nabla^\mu R\nabla_\mu R \\
& +\left(-6a(d-1) + \frac{(d+4)b}{2}\right)R_{\kappa\lambda}R^{\kappa\lambda}R^2 \\
& \left.+(d-1)\left(12a + 2b + \frac{e}{2} - 2f\right)R_{\kappa\lambda}\nabla^\kappa R\nabla^\lambda R\right\}
\end{aligned}$$



$$\begin{aligned}
& + \left( 12a(d-1) + b(d-2) + \frac{de}{2} \right) R_{\kappa\lambda} R \nabla^\kappa \nabla^\lambda R \\
& + \left( -2b(d-1) - 2e(d-1) + \frac{dg}{2} \right) R \nabla_\alpha R_{\mu\nu} \nabla^\alpha R^{\mu\nu} \\
& + (-4b(d-1) - e(d-2) - 2g) R R^{\mu\nu} \nabla^2 R_{\mu\nu} \\
& + 2(d-1) \left\{ -b \left( R_{\kappa\lambda} R^{\kappa\lambda} \right)^2 + (2b+e) R_{\kappa\lambda} R^{\mu\nu} \nabla^\kappa \nabla^\lambda R_{\mu\nu} \right. \\
& + (2b+2e-g) R_{\kappa\lambda} \nabla^\kappa R_{\mu\nu} \nabla^\lambda R^{\mu\nu} \\
& + (2b+e+g) R_{\kappa\lambda} R^{\mu\kappa} \nabla_\mu \nabla^\lambda R + (2b-g) R_{\kappa\lambda} \nabla_\mu R \nabla^\lambda R^{\kappa\mu} \left. \right\} \\
& + \left( -4b(d-1) + \frac{(6-d)e}{2} + 2g \right) R R_{\kappa\lambda} R_{\alpha\mu} R^{\alpha\kappa\mu\lambda} \\
& + (d-1) \left\{ (-2b+e+4f) R_{\kappa\lambda} R^{\kappa\lambda} \nabla^2 R - 2(2b+e) R_{\kappa\lambda} \nabla_\mu R \nabla^\mu R^{\kappa\lambda} \right\} \\
& + ((2d-1)e - 2g) R \nabla_\nu R^{\mu\rho} \nabla_\rho R_\mu^\nu \\
& + (e(d-1) - 2g) R R^{\mu\rho} R_\mu^\nu R_{\nu\rho} - de R R^{\mu\nu\rho\sigma} \nabla_\mu \nabla_\rho R_{\nu\sigma} \\
& + 4(d-1) \left\{ -(e+g) R_{\kappa\lambda} R^{\lambda\nu\rho\sigma} R_\rho^\kappa R_{\nu\sigma} + e R_{\kappa\lambda} R_{\sigma\mu\nu}^\kappa \nabla^\mu \nabla^\lambda R^{\nu\sigma} \right. \\
& + (-e+g) R_{\kappa\lambda} \nabla^\lambda R^{\mu\nu} \nabla_\mu R_\nu^\kappa + e R_{\kappa\lambda} \nabla^\mu R^{\nu\sigma} \nabla^\lambda R_{\sigma\mu\nu}^\kappa \\
& - (e+g) R_{\kappa\lambda} R^{\nu\sigma} \nabla^\lambda \nabla_\sigma R_\nu^\kappa \\
& + e \left( R_{\kappa\lambda} R^{\nu\sigma} R^{\alpha\kappa\lambda\mu} R_{\alpha\sigma\mu\nu} + R_{\kappa\lambda} R_\sigma^\nu R^{\alpha\sigma\lambda\mu} R_{\alpha\mu\nu}^\kappa + R_{\kappa\lambda} R_\nu^\sigma R^{\alpha\nu\lambda\mu} R_{\sigma\mu\alpha}^\kappa \right) \left. \right\} \\
& + 2(d-1)(-e+2g) \left\{ R_\kappa^\rho R_{\nu\rho\sigma}^\kappa \nabla^2 R^{\nu\sigma} + 2R_\kappa^\rho \nabla^\mu R^{\nu\sigma} \nabla_\mu R_{\nu\rho\sigma}^\kappa \right. \\
& + R_\kappa^\rho R^{\nu\sigma} \nabla^2 R_{\nu\rho\sigma}^\kappa \left. \right\} + \left( 2f(d-1) + \frac{g(d-2)}{2} \right) R \nabla^4 R \\
& + 2(d-1) \left\{ (e+2g) R_{\kappa\lambda} R^{\lambda\rho} R^{\kappa\alpha} R_{\alpha\rho} \right. \\
& + e R_{\kappa\lambda} \nabla_\nu R^{\lambda\mu} \nabla_\mu R^{\nu\kappa} - (2f+g) R_{\kappa\lambda} \nabla^\kappa \nabla^\lambda \nabla^2 R \\
& + g \left\{ -4R_{\kappa\lambda} \nabla_\mu R_\rho^\kappa \nabla^\mu R^{\lambda\rho} - 2R_{\kappa\lambda} R^{\kappa\nu} \nabla^2 R_\nu^\lambda + R_{\kappa\lambda} \nabla^4 R^{\kappa\lambda} \right\} \left. \right\} \quad (\text{E.26})
\end{aligned}$$

The contributions from the combination  $[\mathcal{L}_{loc}]_4 = X R^2 + Y R^{\mu\nu} R_{\mu\nu}$  and  $[\mathcal{L}_{loc}]_4 = X R^2 + Y R^{\mu\nu} R_{\mu\nu}$  are also considered. From the combination of  $\mathcal{L}_1 = X R^2$  and  $\mathcal{L}_2 = X R^2$ , we get

$$\begin{aligned}
\frac{1}{\sqrt{G}} [\{S_1, S_2\}]_{wt=8} &= \frac{-1}{d-1} G_{\mu\nu} G_{\kappa\lambda} \frac{\delta S_1}{G_{\mu\nu}} \frac{\delta S_2}{G_{\kappa\lambda}} - G_{\mu\kappa} G_{\nu\lambda} \frac{\delta S_1}{G_{\mu\nu}} \frac{\delta S_2}{G_{\kappa\lambda}} \\
&= X^2 \left[ \frac{-1}{d-1} \left\{ \left( \frac{d}{2} - 2 \right) R^2 + 2(1-d) \nabla^2 R \right\}^2 \right. \\
&\quad \left. + \left\{ \frac{1}{2} R^2 G_{\kappa\lambda} + 2 \left( -R R_{\kappa\lambda} + \nabla_\kappa \nabla_\lambda R - G_{\kappa\lambda} \nabla^2 R \right) \right\} \right]
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ \frac{1}{2} R^2 G^{\kappa\lambda} + 2 \left( -R R^{\kappa\lambda} + \nabla^\kappa \nabla^\lambda R - G^{\kappa\lambda} \nabla^2 R \right) \right\} \Big] \\
& = X^2 \left[ \frac{-1}{d-1} \left\{ \left( \frac{d}{2} - 2 \right)^2 R^4 + 4(1-d)^2 (\nabla^2 R)^2 \right. \right. \\
& \quad + 4 \left( \frac{d}{2} - 2 \right) (1-d) R^2 \nabla^2 R \Big\} + d \left( \frac{1}{2} R^2 - 2 \nabla^2 R \right)^2 \\
& \quad - 4 R^2 \left( \frac{1}{2} R^2 - 2 \nabla^2 R \right) + 4 \nabla^2 R \left( \frac{1}{2} R^2 - 2 \nabla^2 R \right) \\
& \quad + 4 R^{\kappa\lambda} R_{\kappa\lambda} R^2 - 8 R R^{\kappa\lambda} \nabla_\kappa \nabla_\lambda R + 4 \nabla_\kappa \nabla_\lambda R \nabla^\kappa \nabla^\lambda R \Big] \\
& = X^2 \left[ -\frac{d+8}{4(d-1)} R^4 + 2 R^2 \nabla^2 R - 4 (\nabla^2 R)^2 \right. \\
& \quad \left. + 4 R^2 R^{\mu\nu} R_{\mu\nu} + 4 \nabla^\kappa \nabla^\lambda R \nabla_\kappa \nabla_\lambda R - 8 R R^{\kappa\lambda} \nabla_\kappa \nabla_\lambda R \right]. \quad (\text{E.27})
\end{aligned}$$

From the combination of  $\mathcal{L}_1 = Y R^{\mu\nu} R_{\mu\nu}$  and  $\mathcal{L}_2 = Y R^{\mu\nu} R_{\mu\nu}$ , we get

$$\begin{aligned}
\frac{1}{\sqrt{G}} [\{S_1, S_2\}]_{wt=8} & = Y^2 \left[ \frac{-1}{d-1} \left\{ \left( \frac{d}{2} - 2 \right) R^{\mu\nu} R_{\mu\nu} - \frac{d}{2} \nabla^2 R \right\}^2 \right. \\
& \quad + \left\{ \frac{1}{2} R^{\mu\nu} R_{\mu\nu} G^{\kappa\lambda} + \nabla_\kappa \nabla_\lambda R - 2 R_{\kappa\mu\lambda\nu} R^{\mu\nu} - \nabla^2 R_{\kappa\lambda} - \frac{1}{2} G^{\kappa\lambda} \nabla^2 R \right\} \\
& \quad \times \left\{ \frac{1}{2} R^{\mu\nu} R_{\mu\nu} G^{\kappa\lambda} - 2 R_{\mu\nu} R^{\kappa\nu} G^{\mu\lambda} + \nabla^\kappa \nabla^\lambda R - 2 G^{\kappa\omega} R_{\mu\omega\nu}^\lambda R^{\mu\nu} \right. \\
& \quad \left. + 2 R_{\mu\nu} R^{\lambda\nu} G^{\mu\kappa} - \nabla^2 R^{\kappa\lambda} - \frac{1}{2} G^{\kappa\lambda} \nabla^2 R \right\} \\
& = Y^2 \left[ \frac{-1}{d-1} \left\{ \left( \frac{d}{2} - 2 \right)^2 (R^{\mu\nu} R_{\mu\nu})^2 + \frac{d^2}{4} (\nabla^2 R)^2 \right. \right. \\
& \quad + d \left( \frac{d}{2} - 2 \right) R^{\mu\nu} R_{\mu\nu} \nabla^2 R \Big\} + \frac{d}{4} (R^{\mu\nu} R_{\mu\nu})^2 - (R_{\mu\nu} R^{\mu\nu})^2 \\
& \quad + \frac{1}{2} R^{\mu\nu} R_{\mu\nu} \nabla^2 R - (R^{\mu\nu} R_{\mu\nu})^2 \\
& \quad + (R^{\mu\nu} R_{\mu\nu})^2 - \frac{1}{2} R^{\mu\nu} R_{\mu\nu} \nabla^2 R - \frac{d}{4} R^{\mu\nu} R_{\mu\nu} \nabla^2 R \\
& \quad + \frac{1}{2} R^{\mu\nu} R_{\mu\nu} \nabla^2 R - 2 R_{\mu\nu} R^{\kappa\nu} \nabla_\kappa \nabla^\mu R + \nabla^\kappa \nabla^\lambda R \nabla_\kappa \nabla_\lambda R \\
& \quad - 2 R_{\mu\omega\nu}^\lambda R^{\mu\nu} \nabla^\omega \nabla_\lambda R + 2 R_{\mu\nu} R^{\lambda\nu} \nabla^\mu \nabla_\lambda R - \nabla^2 R^{\kappa\lambda} \nabla_\kappa \nabla_\lambda R \\
& \quad - \frac{1}{2} (\nabla^2 R)^2 - (R^{\mu\nu} R_{\mu\nu})^2 + 4 R_{\mu\nu} R^{\kappa\nu} R_{\beta\kappa\alpha}^\mu R^{\alpha\beta} \\
& \quad - 2 \nabla^\kappa \nabla^\lambda R R_{\kappa\mu\lambda\nu} R^{\mu\nu} + 4 R_{\mu\omega\nu}^\lambda R^{\mu\nu} R_{\alpha\lambda\beta}^\omega R^{\alpha\beta} - 4 R_\nu^\kappa R^{\lambda\nu} R_{\kappa\alpha\lambda\beta} R^{\alpha\beta} \\
& \quad + 2 \nabla^2 R^{\kappa\lambda} R_{\kappa\mu\lambda\nu} R^{\mu\nu} + R_{\mu\nu} R^{\mu\nu} \nabla^2 R \\
& \quad \left. - \frac{1}{2} R^{\mu\nu} R_{\mu\nu} \nabla^2 R + 2 R_{\mu\nu} R^{\kappa\nu} \nabla^2 R_\kappa^\mu - \nabla^\kappa \nabla^\lambda R \nabla^2 R_{\kappa\lambda} \right]
\end{aligned}$$

$$\begin{aligned}
& +2R_{\mu\omega\nu}^{\lambda}R^{\mu\nu}\nabla^2R_{\lambda}^{\omega}-2R_{\mu\nu}R^{\lambda\nu}\nabla^2R_{\lambda}^{\mu}+\nabla^2R^{\kappa\lambda}\nabla^2R_{\kappa\lambda}+\frac{1}{2}\left(\nabla^2R\right)^2 \\
& -\frac{d}{4}R^{\mu\nu}R_{\mu\nu}\nabla^2R+R_{\mu\nu}R^{\mu\nu}\nabla^2R-\frac{1}{2}\left(\nabla^2R\right)^2 \\
& +R_{\mu\nu}R^{\mu\nu}\nabla^2R-R_{\mu\nu}R^{\mu\nu}\nabla^2R+\frac{1}{2}\left(\nabla^2R\right)^2+\frac{d}{4}\left(\nabla^2R\right)^2\Big] \\
= & Y^2\left[-\frac{d+8}{4(d-1)}\left(R^{\mu\nu}R_{\mu\nu}\right)^2-\frac{d}{4(d-1)}\left(\nabla^2R\right)^2\right. \\
& +\frac{(d-4)(-2d+1)}{2(d-1)}R^{\mu\nu}R_{\mu\nu}\nabla^2R+\nabla^{\kappa}\nabla^{\lambda}R\nabla_{\kappa}\nabla_{\lambda}R \\
& -4R_{\mu\omega\nu}^{\lambda}R^{\mu\nu}\nabla^{\omega}\nabla_{\lambda}R-2\nabla^2R^{\kappa\lambda}\nabla_{\kappa}\nabla_{\lambda}R+4R_{\mu\omega\nu}^{\lambda}R^{\mu\nu}R_{\alpha\lambda\beta}^{\omega}R^{\alpha\beta} \\
& \left.+4R_{\kappa\mu\lambda\nu}R^{\mu\nu}\nabla^2R^{\kappa\lambda}+\nabla^2R^{\kappa\lambda}\nabla^2R_{\kappa\lambda}\right] \tag{E.28}
\end{aligned}$$

From the combination of  $\mathcal{L}_1 = XR^2$  and  $\mathcal{L}_2 = YR^{\mu\nu}R_{\mu\nu}$ , we get

$$\begin{aligned}
\frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=8} &= XY\left[\frac{-1}{d-1}\left\{\left(\frac{d}{2}-2\right)R^2+2(1-d)\nabla^2R\right\}\right. \\
&\times\left\{\left(\frac{d}{2}-2\right)R^{\mu\nu}R_{\mu\nu}-\frac{d}{2}\nabla^2R\right\} \\
&+\left\{\frac{1}{2}R^2G_{\kappa\lambda}+2\left(-RR_{\kappa\lambda}+\nabla_{\kappa}\nabla_{\lambda}R-G_{\kappa\lambda}\nabla^2R\right)\right\} \\
&\times\left\{\frac{1}{2}R^{\mu\nu}R_{\mu\nu}G^{\kappa\lambda}+\nabla^{\kappa}\nabla^{\lambda}R-2G^{\kappa\omega}R_{\mu\omega\nu}^{\lambda}R^{\mu\nu}\right. \\
&\left.-\nabla^2R^{\kappa\lambda}-\frac{1}{2}G^{\kappa\lambda}\nabla^2R\right\}\Big] \\
= & XY\left[-\frac{d+8}{4(d-1)}R^2R^{\mu\nu}R_{\mu\nu}+\frac{-d+4}{4(d-1)}R^2\nabla^2R\right. \\
& +R^{\mu\nu}R_{\mu\nu}\nabla^2R-\left(\nabla^2R\right)^2-2RR_{\kappa\lambda}\nabla^{\kappa}\nabla^{\lambda}R \\
& +4R_{\mu\omega\nu}^{\lambda}R^{\mu\nu}RR_{\lambda}^{\omega}+2\nabla^2R^{\kappa\lambda}R_{\kappa\lambda}R+2\nabla^{\kappa}\nabla^{\lambda}R\nabla_{\kappa}\nabla_{\lambda}R \\
& \left.-4R_{\mu\omega\nu}^{\lambda}R^{\mu\nu}\nabla^{\omega}\nabla_{\lambda}R-2\nabla^2R^{\kappa\lambda}\nabla_{\kappa}\nabla_{\lambda}R\right] \tag{E.29}
\end{aligned}$$

Thus the contribution from  $\mathcal{L}_1 = XR^2 + YR^{\mu\nu}R_{\mu\nu}$  and  $\mathcal{L}_2 = XR^2 + YR^{\mu\nu}R_{\mu\nu}$  for the calculations of  $[\{S_1, S_2\}]_{wt=8}$  are following form.

$$\begin{aligned}
\frac{1}{\sqrt{G}}[\{S_1, S_2\}]_{wt=8} &= X^2\left[-\frac{d+8}{4(d-1)}R^4+2R^2\nabla^2R-4\left(\nabla^2R\right)^2\right. \\
& \left.+4R^2R^{\mu\nu}R_{\mu\nu}+4\nabla^{\kappa}\nabla^{\lambda}R\nabla_{\kappa}\nabla_{\lambda}R-8RR^{\kappa\lambda}\nabla_{\kappa}\nabla_{\lambda}R\right] \\
& +Y^2\left[-\frac{d+8}{4(d-1)}\left(R^{\mu\nu}R_{\mu\nu}\right)^2-\frac{d}{4(d-1)}\left(\nabla^2R\right)^2\right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{(d-4)(-2d+1)}{2(d-1)} R^{\mu\nu} R_{\mu\nu} \nabla^2 R + \nabla^\kappa \nabla^\lambda R \nabla_\kappa \nabla_\lambda R \\
& - 4R_{\mu\omega\nu}^\lambda R^{\mu\nu} \nabla^\omega \nabla_\lambda R - 2\nabla^2 R^{\kappa\lambda} \nabla_\kappa \nabla_\lambda R + 4R_{\mu\omega\nu}^\lambda R^{\mu\nu} R_{\alpha\lambda\beta}^\omega R^{\alpha\beta} \\
& + 4R_{\kappa\mu\lambda\nu} R^{\mu\nu} \nabla^2 R^{\kappa\lambda} + \nabla^2 R^{\kappa\lambda} \nabla^2 R_{\kappa\lambda} \Big] \\
& + 2XY \Big[ - \frac{d+8}{4(d-1)} R^2 R^{\mu\nu} R_{\mu\nu} + \frac{-d+4}{4(d-1)} R^2 \nabla^2 R \\
& + R^{\mu\nu} R_{\mu\nu} \nabla^2 R - \left( \nabla^2 R \right)^2 - 2R R_{\kappa\lambda} \nabla^\kappa \nabla^\lambda R \\
& + 4R_{\mu\omega\nu}^\lambda R^{\mu\nu} R R_\lambda^\omega + 2\nabla^2 R^{\kappa\lambda} R_{\kappa\lambda} R + 2\nabla^\kappa \nabla^\lambda R \nabla_\kappa \nabla_\lambda R \\
& - 4R_{\mu\omega\nu}^\lambda R^{\mu\nu} \nabla^\omega \nabla_\lambda R - 2\nabla^2 R^{\kappa\lambda} \nabla_\kappa \nabla_\lambda R \Big] \\
= & - \frac{(d+8)X^2}{4(d-1)} R^4 + \left( 2X^2 + \frac{(-d+4)XY}{2(d-1)} \right) R^2 \nabla^2 R \\
& + \left( -4X^2 - \frac{dY^2}{4(d-1)} - 2XY \right) (\nabla^2 R)^2 \\
& + \left( 4X^2 - \frac{(d+8)XY}{2(d-1)} \right) R^2 R^{\mu\nu} R_{\mu\nu} - 4(2X^2 + XY) R R^{\kappa\lambda} \nabla_\kappa \nabla_\lambda R \\
& + (4X^2 + Y^2 + 4XY) \nabla^\kappa \nabla^\lambda R \nabla_\kappa \nabla_\lambda R \\
& - \frac{(d+8)Y^2}{4(d-1)} (R^{\mu\nu} R_{\mu\nu})^2 - 4(Y^2 + 2XY) R_{\mu\omega\nu}^\lambda R^{\mu\nu} \nabla^\omega \nabla_\lambda R \\
& + \left( \frac{(d-4)(-2d+1)}{2(d-1)} Y^2 + 2XY \right) R^{\mu\nu} R_{\mu\nu} \nabla^2 R \\
& - 2(Y^2 + 2XY) \nabla^2 R^{\kappa\lambda} \nabla_\kappa \nabla_\lambda R + 4Y^2 R_{\mu\omega\nu}^\lambda R^{\mu\nu} R_{\alpha\lambda\beta}^\omega R^{\alpha\beta} \\
& + 4Y^2 R_{\kappa\mu\lambda\nu} R^{\mu\nu} \nabla^2 R^{\kappa\lambda} + Y^2 \nabla^2 R^{\kappa\lambda} \nabla^2 R_{\kappa\lambda} \\
& + 8XY R_{\mu\omega\nu}^\lambda R^{\mu\nu} R R_\lambda^\omega + 4XY \nabla^2 R^{\kappa\lambda} R_{\kappa\lambda} R
\end{aligned} \tag{E.30}$$

Summing up this result (E.30) and the previous result (E.26), we reproduce the  $[\{S_1, S_2\}]_{wt=8}$  (without contributions from  $[\mathcal{L}_8]$ ) as the result (8.28).

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